

# An abelian category of motivic sheaves

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## Abstract

A category of motivic “sheaves” is constructed over a variety in characteristic 0 using Nori’s method. Although the relationship with many alternative constructions remains to be clarified, it does have many of the properties one expects. For example, it is abelian and has Betti, Hodge and  $\ell$ -adic realizations, and it has a Tannakian subcategory of motivic local systems.

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## 1 Introduction

The basic homological invariants of a fibration of topological spaces  $f : X \rightarrow S$ , are the local systems  $R^i f_* \mathbb{Q}$ . When this is a family of complex algebraic varieties defined over a subfield  $k$  of  $\mathbb{C}$ , there are many related invariants, such as the Gauss-Manin connection, the associated variation of mixed Hodge structure, and the action of the algebraic fundamental group on étale cohomology of the fibres. According to Grothendieck’s philosophy, all of these structures should come from the motive of the family. My goal here is to make this idea precise in the following way. Given a field  $F$ , and a variety  $S$ , as above, I will construct an abelian category  $\mathcal{M}(S; F)$  of motivic “sheaves” of  $F$ -modules. The above local systems can be promoted to objects in  $\mathcal{M}(S; \mathbb{Q})$ , and the associated structures can be obtained by applying appropriate realizations functors.

Before explaining what I will do, let me say a few words about what I won’t. The usual approach to building a category of motives is to start with a category of varieties and algebraic correspondences and modify and complete this in some way. This stays very close to the underlying geometry which is good. On the other hand, it is usually very hard to prove for example that what one gets is (derived from) an abelian category. A more pragmatic approach is to take a system of compatible realizations. This usually has good categorical properties, but is somewhat ad hoc in nature; and in the relative setting, it would appear that any such approach would be necessarily very technical (e.g. [S3]).

Here I want to take a middle path first blazed by Nori while building a category of motives over a field [N2, L2]. The approach appeals to a particular

realization at the outset, but is essentially geometric in its character. Since the construction is not that widely known, I will indicate the basic idea starting with a toy model and then refining it. In fact, one of the goals of this paper is to give an exposition of some, although not all, aspects of Nori's construction. Consider the category of  $k$ -algebraic varieties  $Var_k$ . Since we assume that  $k \subseteq \mathbb{C}$ , we may apply singular (or Betti) cohomology  $H^i$  to obtain a contravariant functor from  $Var_k$  to the category  $\mathbb{Q}\text{-mod}$  of finite dimensional  $\mathbb{Q}$ -vector spaces. The key point is that  $H^i(X)$  is not just a vector space, but a module over the ring of natural transformations  $End(H^i)$ , or a comodule over the "dual" coalgebra  $End^\vee(H^i)$  (section 2.1) which is technically better behaved. The category  $\mathcal{M}'_i(k)$  of finite dimensional comodules over this coalgebra forms an approximation to Nori's category. It is abelian, and the objects  $M \in \mathcal{M}'_i(k)$  are not too wild, in that they admit presentations of the form

$$\bigoplus_j H^i(X_j) \rightarrow \bigoplus_k H^i(Y_k) \rightarrow M \rightarrow 0$$

Furthermore, each object  $M$  also carries a canonical mixed Hodge structure and (after tensoring with  $\mathbb{Q}_\ell$ ) an action of  $Gal(\bar{k}/k)$  as one would hope. So far so good, but it would be better to include the various  $\mathcal{M}'_i$  into a single category  $\mathcal{M}$ , so that standard exact sequences respect the  $\mathcal{M}$ -structure. Toward this end, it is necessary to modify the basic construction by incorporating boundary operators into the foundations. Thus instead of starting with  $Var_k$ , the source category  $\Delta$  consists of triples  $(X, Y \subset X, i \in \mathbb{N})$  and the appropriate notion of morphism, which includes abstract boundary maps  $(X, Y, i) \rightarrow (Y, \emptyset, i-1)$ . This is really a partial category in the sense that the composition law is only partially defined; nevertheless the basic constructions go through. The category of comodules over  $End^\vee(H)$ , where  $H : \Delta \rightarrow \mathbb{Q}\text{-mod}$  is the functor sending  $(X, Y, i) \mapsto H^i(X, Y)$ , yields a rational version of Nori's category of effective cohomological motives. Following the usual pattern, the category  $\mathcal{M}(k)$  is obtained by inverting the Tate motive. This step can be built in from the beginning, and we find it convenient to do so.

Now turning to the general case, the building blocks for  $\mathcal{M}(S; F)$  are quadruples consisting of a quasiprojective family  $f : X \rightarrow S$ , a closed subvariety  $Y \subset X$  and indices  $i \in \mathbb{N}, w \in \mathbb{Z}$ . This is subject to a further technical admissibility condition (definition 3.2.1) which will be satisfied if  $f$  is projective. When  $Y = \emptyset$ , this data represents the motivic version of  $R^i f_* F(w)$  denoted here by  $h_S^i(X)(w)$ . The parameter  $w$  keeps track of Tate twists, which although extraneous for ordinary sheaves are nontrivial in the Hodge and étale realizations. For nonempty  $Y$ , the associated motive  $h_S^i(X, Y)(w)$  roughly corresponds to the fiberwise cohomology of the pair. In essence,  $\mathcal{M}(S; F)$  is set up as the universal theory for which:

- (M1)  $\mathcal{M}(S; F)$  is an  $F$ -linear abelian category with a faithful exact functor  $R_B$  to the category of sheaves of  $F$ -modules on  $S$  with its classical topology.
- (M2) A morphism  $X' \rightarrow X$  over  $S$ , taking  $Y'$  to  $Y$  would give rise to a morphism

of  $h_S^i(X, Y)(w) \rightarrow h_S^i(X', Y')(w)$  compatible with the usual pullback map under  $R_B$ .

- (M3) Whenever  $Z \subseteq Y \subseteq X$ , there are connecting morphisms  $h_S^i(X, Y)(w) \rightarrow h_S^{i+1}(Y, Z)(w)$  compatible with the usual pullback maps.
- (M4)  $h_S^{i+2}(X \times \mathbb{P}^1, X \times \{0\} \cup Y \times \mathbb{P}^1)(w) \cong h_S^i(X, Y)(w - 1)$ .
- (M5) Objects and morphisms of  $\mathcal{M}(S; F)$  can be patched on a Zariski open cover.

The actual construction is obtained by modifying the framework discussed in the previous paragraph. Given stratification  $\Sigma$  and a collection of base points on the strata, let  $\Delta(\Sigma)$  be the collection of quadruples such that the cohomology sheaf is constructible with respect to  $\Sigma$ . We can make this into a partial category by adding morphisms corresponding to items (M2), (M3) and (M4). The functor  $H_\Sigma : \Delta(\Sigma) \rightarrow \mathbb{Q}\text{-mod}$  is defined by sending  $(X, Y, i, w)$  to the product  $H^i(X_s, Y_s)$  at the given set of base points. The category  $\mathcal{PM}(S, \Sigma; F)$  of  $\Sigma$ -constructible premotivic sheaves is constructed explicitly as the category of comodules over  $\text{End}^\vee(H_\Sigma)$ . Note that contrary to initial appearances, this is not simply a product of  $\mathcal{M}(k)$  over the base points because  $\Delta(\Sigma)$  does not decompose this way (see example 3.5.3). The trivial exception is when  $S$  is a finite set of points. The category  $\mathcal{PM}(S; F)$  of premotivic sheaves is given as the direct limit of these categories as  $\Sigma$  gets finer. This will satisfy (M1) to (M4). The category  $\mathcal{M}(S; F)$  is obtained from  $\mathcal{PM}(S; F)$  by forcing (M5) by passing to the associated stack. This last step can be made explicit. In fact a weak form of (M5) holds for  $\mathcal{PM}$ . So it is not quite clear to me whether this axiom is redundant, nevertheless it is included for completeness.

Here are the precise properties:

**Theorem 1.0.1.** *To every  $k$ -variety, there is an  $F$ -linear abelian category  $\mathcal{M}(S; F)$  such that:*

1. *These are defined over the prime field  $F_0$ , i.e.  $\mathcal{M}(S, F) \cong \mathcal{M}(S, F_0) \otimes_{F_0} F$ .*
2. *There is an exact Betti realization functor*

$$R_B : \mathcal{M}(S; F) \rightarrow \text{Cons}(S_{an}, F)$$

*to the category of constructible sheaves of  $F$ -modules for the classical topology.*

3. *There is an exact Hodge realization functor*

$$R_H : \mathcal{M}(S; \mathbb{Q}) \rightarrow \text{Cons-MHM}(S) \subset D^b \text{MHM}(S)$$

*to the heart of the classical  $t$ -structure of the derived category mixed Hodge modules (see appendix C).*

4. There is an exact étale realization functor

$$R_{\text{ét}} : \mathcal{M}(S; F) \rightarrow \text{Cons}(S_{\text{ét}}, F)$$

to the category of constructible sheaves of  $F$ -modules for the étale topology. (In this case,  $F$  should be finite or  $\mathbb{Q}_\ell$ .)

5. When  $f$  satisfies a suitable admissibility condition (of being controlled in the sense of definition 3.2.1), there exist motives in  $\mathcal{M}(S; F)$  corresponding to  $R^i f_* F(n)$  under realization.
6. There are inverse images compatible with realizations.
7. There are higher direct images for projective or constant maps compatible with realizations.

Many of the above items are formal consequences of the definitions, but the last is not. The construction of direct images is technically the most difficult part of this paper. General arguments give the existence of an adjoint to inverse image which ought to play the role of the direct image. Proving that this has reasonable properties for projective maps requires work, which uses a refinement of the method of [Ar]. This earlier paper was really the starting point for this entire project. This ultimately hinges on Nori's insight that Beilinson's "basic lemma" can be used to construct cell decompositions which reduce the homological complexities. In the relative setting, there are few additional complications. For instance, these decompositions are only obtained locally over the base, so patching issues of the sort given in (M5) comes into play. But modulo these technicalities, the basic strategy of using cell decompositions does work.

The objects of  $\mathcal{M}(S)$  play the role of constructible sheaves. Inside this, we have a subcategory of "local systems" arising from particularly nice families  $(X \rightarrow S, Y, i, w)$ . The precise condition is that  $X$  can be completed to a smooth projective map so that  $Y$  together with the boundary is a divisor with relative normal crossings. These enjoy the following good properties.

**Theorem 1.0.2.** *There is an abelian full subcategory  $\mathcal{M}_{l_s}(S; \mathbb{Q}) \subset \mathcal{M}(S, \mathbb{Q})$  of motivic local systems such that:*

1. The images of  $\mathcal{M}_{l_s}(S; \mathbb{Q})$  ( respectively  $\mathcal{M}_{l_s}(S; \mathbb{Q}_\ell)$ ) under  $R_B$  ( respectively  $R_{\text{ét}}$ ) is contained in the category of locally constant ( respectively lisse sheaves). The image under  $R_H$  is contained in the category  $\text{VMHS}(S_{\text{an}})$  of admissible variations of mixed Hodge structures.
2. There are tensor products on  $\mathcal{M}_{l_s}(S; F)$  compatible with realizations. With this structure it is a Tannakian category.
3. The subcategory  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q}) \subset \mathcal{M}_{l_s}(S, \mathbb{Q})$  generated by smooth projective families is a semisimple Tannakian category.
4. Objects in  $\mathcal{M}_{l_s}(S, \mathbb{Q})$  carry a weight filtration such that the associated graded objects are pure.

A number of the arguments again rely on the existence of cellular decompositions. Regarding item 4, I do not have a good notion of weight in  $\mathcal{M}(S)$  at present. I expect that it would require the development of an analogue of perverse sheaf in  $D^b\mathcal{M}(S)$ , since the pure objects are almost surely of this form.

A natural question, that is only partially solved here, is the relation of this approach to motives to the others. André [A1] defines the class of motivated cycles on smooth projective variety to be the cycles which would be expected to be algebraic assuming Grothendieck’s standard conjectures. André showed that the category of pure motives over a field constructed with such correspondences has all the expected properties. This construction can be extended to more general bases without much difficulty [AD]. We show that this category is precisely  $M_{pure}(S)$ . In his unpublished work, Nori has constructed a functor from Voevodsky’s category  $D_{gm}(k)$  to  $D^b\mathcal{M}(k)$ . It seems reasonable to expect that this generalizes over a base, but such matters will be postponed for the future. In the final section, I discuss Nori’s Hodge conjecture which says that  $\mathcal{M}(\mathbb{C})$  embeds fully faithfully into the category of mixed Hodge structures. This would imply that the canonical mixed Hodge structure on cohomology is “Galois invariant”. The relative case can be reduced to this by rather formal argument involving direct image and restriction functors.

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*Notation:* Since the notation will tend to get rather heavy, *I will routinely suppress subscripts, superscripts and others symbols whenever they can be understood from context.* Given a ring  $R$ , let  $R\text{-Mod}$  (respectively  $R\text{-mod}$ ) stand for the category of (finitely presented) left  $R$ -modules. Fix a field  $k$  embeddable into  $\mathbb{C}$  and another field  $F$ . For most of the paper, I will work with a fixed embedding  $\iota : k \hookrightarrow \mathbb{C}$ . A  $k$ -variety is simply a reduced separated  $k$ -scheme of finite type. Let  $Var_k$  be the category of these. Given a  $k$ -variety  $X$ , the word point  $x \in X$  generally refers to a  $\bar{k}$ -rational point. I will denote the analytic space  $(X \times_{\iota} Spec \mathbb{C})_{an}$  by  $X_{\iota,an}$  or  $X_{an}$  or sometimes just  $X$ , in keeping with the previous comment regarding notation. A quasi-projective morphism is a morphism which can be factored as a composition of an open immersion followed by a projective morphism. I will usually write  $H^i(X; F)$  for  $H^i(X_{\iota,an}; F)$ . Given a map  $f : X \rightarrow S$  of spaces and a sheaf  $\mathcal{F}$  on  $X$ , I will often denote the higher direct image  $R^i f_* \mathcal{F}$  by  $H_S^i(X, \mathcal{F})$ . Since this will *never* be used to denote cohomology with support in this paper, there should be no danger of confusion.

## 2 Representations of graphs

### 2.1 Endomorphism coalgebras

In the next couple of sections, we set up the basic foundation for the rest of the paper. Let  $F$  be a field. Suppose we are given an  $F$ -linear abelian category  $\mathcal{A}$  with an exact faithful embedding  $H$  into the category of finite dimensional vector spaces  $F\text{-mod}$ . Then the ring  $End(H) = End_F(H)$ , of  $F$ -linear natural transformations of  $H$  to itself, will act naturally on  $H(A)$  for any  $A \in Ob\mathcal{A}$ . This suggests that one might be able to reconstruct  $\mathcal{A}$  as the category of finite dimensional modules over this ring. However, this does not generally work (example 2.2.8). The right thing to consider is the category of comodules over the dual object  $End^\vee(H)$  whose construction we learned from [JS]. Before getting into the construction, we should explain how to characterize it. Given a commutative  $F$ -algebra  $R$ , we can form new category  $\mathcal{A} \otimes R$  with the same objects as  $\mathcal{A}$ , but  $Hom_{\mathcal{A} \otimes R}(-, -) = Hom_{\mathcal{A}}(-, -) \otimes R$ . The functor  $H$  extends to an  $R$ -linear functor  $H \otimes R : \mathcal{A} \otimes R \rightarrow R\text{-mod}$ . In this way, we have an algebra valued functor  $R \mapsto End_R(H \otimes R)$ .

**Lemma 2.1.1.** *This functor is represented by a colgebra  $End^\vee(H)$ , i.e.*

$$Hom_F(End^\vee(H), R) \cong End_R(H \otimes R)$$

This implies that  $End^\vee(H)^* = End(H)$ , but usually  $End(H)^* \neq End^\vee(H)$ . Nevertheless, most of the statements become easier to follow if one formulates them for  $End(H)$  and dualizes. The lemma tells us how to make sense of this. Note that we can express  $End^\vee(H)$  or any coalgebra as a directed union of finite dimensional subcoalgebras  $\cup E_i$ . Thus the correct dual object to  $End^\vee(H)$  is not  $End(H)$  but the pro-algebra “ $\varprojlim$ ”  $E_i^*$ . Moreover,  $\mathcal{A}$  can be described as 2-colimit of the categories of  $E_i$ -modules. We will find this viewpoint convenient later on, but for the moment, it seems simpler to work with the coalgebra.

Given pair of functors  $G, H : C \rightarrow D$ , with  $D$   $F$ -linear,  $Hom(G, H)$  is an  $F$ -vector space. More explicitly, we can identify  $Hom(G, H)$  with

$$\ker \left[ \prod_{M \in Ob C} Hom(G(M), H(M)) \longrightarrow \prod_{f: N \rightarrow P \in Mor C} Hom(G(N), H(P)) \right] \quad (1)$$

where the map takes the collection  $(\eta_M)_M$  to  $(H(f) \circ \eta_N - \eta_P \circ G(f))_f$ . Composition makes  $End(H) = Hom(H, H)$  into an algebra as noted above. Following [JS], it is convenient to introduce a smaller “predual” object, which means that  $Hom^\vee(G, H)^* = Hom(G, H)$ .

Let  $F\text{-Lin}$  be the collection of  $F$ -linear abelian categories with finite dimensional  $Hom$ 's. Suppose that we now have a pair of functors  $G, H : C \rightarrow D$ , with  $D \in F\text{-Lin}$ . Define  $Hom^\vee(G, H)$  to be the cokernel of

$$\bigoplus_{f: N \rightarrow P \in C} Hom(G(N), H(P))^* \xrightarrow{S} \bigoplus_{M \in Ob C} Hom(G(M), H(M))^* \quad (2)$$

where the map  $S$  is defined so that  $\text{Hom}^\vee(G, H)^* = \text{Hom}(G, H)$ . More explicitly,  $S$  sends  $\eta_f^* \in \text{Hom}(G(N), H(P))^*$  to  $\eta_N^* \in \text{Hom}(G(N), H(N))^*$  plus  $\eta_P^* \in \text{Hom}(G(P), H(P))^*$  where

$$\begin{aligned}\langle \eta_N^*, \eta_N \rangle &= \langle \eta_f^*, H(f) \circ \eta_N \rangle \\ \langle \eta_P^*, \eta_P \rangle &= -\langle \eta_f^*, \eta_P \circ G(f) \rangle\end{aligned}$$

Upon setting  $\text{End}^\vee(H) := \text{Hom}^\vee(H, H)$ , we see that this satisfies lemma 2.1.1 as a vector space, and we can use this formula to define the colagebra structure. However, it will be useful to describe this more explicitly. The sum of the maps

$$\text{End}(H(M))^* \rightarrow F,$$

dual to the identity, is easily seen to factor through  $\text{End}^\vee(H)$  and this defines the counit

$$1_H^\vee : \text{End}^\vee(H) \rightarrow F$$

Given functors  $G, H, L : C \rightarrow D$  with  $D \in F\text{-Lin}$  we have a comultiplication

$$\circ^\vee : \text{Hom}^\vee(G, L) \rightarrow \text{Hom}^\vee(G, H) \otimes \text{Hom}^\vee(H, L)$$

dual to composition  $\circ$ . More precisely,  $\circ$  is given by product of compositons

$$c_M : \text{Hom}(G(M), H(M)) \otimes \text{Hom}(H(M), L(M)) \rightarrow \text{Hom}(G(M), L(M))$$

Then  $\circ^\vee$  is given by the sum of the dual maps  $c_M^*$

$$\text{Hom}(G(M), L(M))^* \rightarrow \text{Hom}(G(M), H(M))^* \otimes \text{Hom}(H(M), L(M))^* \quad (3)$$

Given  $G, G' : C \rightarrow D$  and  $H, H' : D \rightarrow E$  with  $D, E \in F\text{-Lin}$ , there is composition

$$\diamond^\vee : \text{Hom}^\vee(G' \circ G, H' \circ H) \rightarrow \text{Hom}^\vee(G, H) \otimes \text{Hom}^\vee(G', H')$$

dual to the operation  $\diamond$  defined in appendix A. The operation  $\diamond$  is a product of maps

$$d_M : \text{Hom}(G(M), H(M)) \times \text{Hom}(G'(H(M)), H'(H(M))) \rightarrow \text{Hom}(G' \circ G(M), H' \circ H(M))$$

and  $\diamond^\vee = \sum d_M^*$ . To simplify arguments with these operations, we use the following duality principle:

**Lemma 2.1.2.** *Suppose we are given an identity in  $+, \circ, \diamond, 1_G$ , which amounts to the commutivity of a finite diagram with arrows labelled by these operations. Then the dual identity, obtained by reversing arrows and relabelling by  $+, \circ^\vee, \diamond^\vee, 1_G^\vee$ , also holds.*

*Proof.* Suppose we have a finite diagram with vertices given as finite tensor products of spaces  $\text{Hom}^\vee(-, -)$ , and edges labelled by  $+, \circ^\vee, \diamond^\vee, 1_G^\vee$ . Then commutivity can be established by chasing elements. Given an element of one of the vertices, we can find a subdiagram of finite dimensional vector spaces which contains it. Duality for finite dimensional vector spaces implies that the commutivity of the subdiagram would then follow from commutivity of the dual diagram.  $\square$



Using this principle, we can see that:

**Lemma 2.1.3.** *Given composable functors  $H$  and  $G$*

1.  $End^\vee(G)$  is a colagebra over  $F$  with respect to  $\circ^\vee, 1^\vee$ .
2. The map  $p$  given by

$$\begin{array}{ccc} End^\vee(H \circ G) & \xrightarrow{p} & End^\vee(G) \\ \downarrow \diamond^\vee & \nearrow 1^\vee \otimes 1 & \\ End^\vee(H) \otimes End^\vee(G) & & \end{array}$$

is a colagebra homomorphism.

*Proof.* The first part is clear, since the dual statement is that  $End(G)$  is an algebra. For the second, we have to establish that  $p$  preserves comultiplication. Dually, by identities given in the appendix,

$$\begin{aligned} 1 \diamond (\alpha \circ \beta) &= (1 \circ 1) \diamond (\alpha \circ \beta) \\ &= (1 \diamond \alpha) \circ (1 \diamond \beta) \end{aligned}$$

□

One can now readily verify lemma 2.1.1. We also leave the formulation and proof of the corresponding statement for  $Hom^\vee$  to the reader.

## 2.2 Nori's construction

Any category can be regarded as a directed graph (or diagram in Nori's terminology) by forgetting the composition law. This forgetful functor admits a left adjoint: given a directed graph  $\Delta$ , we can form a category  $Paths(\Delta)$ , whose objects are vertices of  $\Delta$  and morphisms are finite (possibly empty) connected paths between vertices. The adjointness amounts to the obvious fact that given a graph  $\Delta$  and a category  $C$ , there is a one to one correspondence between graph morphisms  $\Delta \rightarrow C$  and functors  $Paths(\Delta) \rightarrow C$ . In view of this, we may apply category theoretic terminology and results to directed graphs.

Let  $H : \Delta \rightarrow F\text{-mod}$  be a functor, i.e. a quiver. We can define  $End^\vee(H)$  by the formula (2), which simplifies to

$$coker\left[ \bigoplus_{f:N \rightarrow P \in Mor \Delta} Hom(H(P), H(N)) \xrightarrow{S} \bigoplus_{M \in Ob \Delta} End(H(M)) \right] \quad (4)$$

where  $S$  takes  $\eta_f \in Hom(H(P), H(N))$  to the difference of  $\eta_f \circ H(f) \in End(H(N))$  and  $H(f) \circ \eta_f \in End(H(P))$ .

We note the following, which is easily checked.

**Lemma 2.2.1.**

1. The collection of functors from graphs to  $F\text{-mod}$  forms a category where the morphisms are commutative diagrams

$$\begin{array}{ccc} \Delta & \xrightarrow{H} & F\text{-mod} \\ \downarrow \pi & \nearrow H' & \\ \Delta' & & \end{array}$$

2.  $\text{End}^\vee(H)$  is isomorphic to  $\text{End}^\vee(\tilde{H})$ , where  $\tilde{H}$  is the extension of  $H$  to  $\text{Paths}(\Delta)$ .
3. The assignment  $(\Delta, H) \mapsto \text{End}^\vee(H)$  is functorial. In particular, there is an induced map  $\text{End}^\vee(H) \rightarrow \text{End}^\vee(H')$  of colagebras where  $H$  and  $H'$  are as in 1.
4. If  $\Delta$  is a category then  $\text{End}^\vee(H) \cong \text{End}^\vee(H')$ , where  $H'$  is the induced functor on the category  $H(\Delta)$  with the same objects as  $\Delta$  but morphisms given by its image under  $H$ .
5. The functor  $(\Delta, H) \mapsto \text{End}^\vee(H)$  preserves finite coproducts. In more explicit terms, if  $\Delta$  decomposes into a disjoint union of  $\Delta_1 \coprod \Delta_2$ , then  $\text{End}^\vee(H) = \text{End}^\vee(H|_{\Delta_1}) \times \text{End}^\vee(H|_{\Delta_2})$  (which is the coproduct of colagebras).

*Proof.* The first statement is clear.

For the second, we have that  $\text{End}^\vee(H)$  and  $\text{End}^\vee(\tilde{H})$  are the quotients of  $\bigoplus \text{End}(H(M))$  by

$$I_H = S\left(\bigoplus_{f \in \text{Mor}\Delta} \text{Hom}(H(P), H(N))\right)$$

and

$$I_{\tilde{H}} = S\left(\bigoplus_{f \in \text{Mor}\text{Paths}(\Delta)} \text{Hom}(H(P), H(N))\right)$$

respectively. Clearly  $I_H \subseteq I_{\tilde{H}}$ . So we have to check the reverse inclusion. We first note that  $S(1) = 0$ , so it suffices to check that  $S(\eta_{f_1 \dots f_n}) \in I_H$  for  $n \geq 2$ . For  $n = 2$ , this follows from the identity

$$S(\eta_{f_1 f_2}) = S(\eta_{f_1 f_2} \circ H(f_1)) + S(H(f_2) \circ \eta_{f_1 f_2}) \in I_H$$

The general case is similar.

Although the third statement is similar to lemma 2.1.3, the previous formalism will not apply without modification. So it is easier to prove directly. An element of  $\text{End}^\vee(H)$  is represented by a finite sum  $\sum h_M$  of elements  $h_M \in \text{End}(H(M))^*$ . Define  $\pi(h_M) = h_{\pi(M)} \in \text{End}(H'(M))^*$ . To see that

this is compatible with comultiplication  $\circ^\vee$ , observe that  $\circ^\vee(h_M) = c_M^*(h_M)$ , where  $c_M^*$  is given in (3). Then

$$\pi(\circ^\vee(\sum_M h_M)) = \pi(\sum_M c_M^*(h_M)) = \sum_M c_{\pi(M)}^*(h_{\pi(M)}) = \circ^\vee(\pi(\sum_M h_M))$$

The fourth and fifth statement follows immediately from the formulas (2) and (4).  $\square$

We let  $End^\vee(H)$ -comod denote the category of right comodules over this coalgebra in  $F$ -mod.

**Corollary 2.2.2.** *A morphism  $(\Delta, H) \rightarrow (\Delta', H')$  as above induces a faithful exact functor  $End^\vee(H)$ -comod  $\rightarrow$   $End^\vee(H')$ -comod.*

*Proof.* This isn't so much a corollary as a statement of the fact that both categories can be viewed as subcategories of  $F$ -mod.  $\square$

We can therefore view  $End^\vee(H)$ -comod as a subcategory of  $End^\vee(H')$ -comod. We will often apply this, without comment, when  $\Delta \subset \Delta'$  is a subgraph and  $H$  is the restriction of  $H'$ .

**Corollary 2.2.3.** *If  $H' : \Delta \rightarrow F$ -mod is another functor with a natural isomorphism  $\Gamma : H \rightarrow H'$ , then  $End^\vee(H)$ -comod and  $End^\vee(H')$ -comod are isomorphic.*

**Corollary 2.2.4.** *Let  $\pi : \tilde{\Delta} \rightarrow \Delta$  be a morphism of graphs such that it is surjective on objects and such that every fiber is connected. Then  $End^\vee(H) \cong End^\vee(H \circ \pi)$ .*

*Proof.* The assumption guarantees that  $H(\text{Paths}(\Delta))$  and  $H \circ \pi(\text{Paths}(\tilde{\Delta}))$  are equivalent.  $\square$

Given  $M \in Ob\Delta$ ,  $H(M)$  is naturally a left  $End(H(M))$ -module, and hence by transpose an  $End(H(M))^*$ -comodule. Via the map  $End(H(M))^* \rightarrow End^\vee(H)$ ,  $M$  becomes a  $End^\vee(H)$ -comodule, which we usually denote by  $h(M)$ . This is a functor  $\Delta \rightarrow End(H)$ -comod. The structure of a general comodule is clarified by the following.

**Lemma 2.2.5.** *Any object  $V$  of  $End^\vee(H)$ -comod fits into an exact sequence*

$$\bigoplus_{i=1}^m h(M_i) \rightarrow \bigoplus_{j=1}^n h(N_j) \rightarrow V \rightarrow 0$$

for some  $M_i, N_j \in Ob\Delta$ .

*Proof.* The lemma will follow from the claim that any comodule is the image of finite sum of the form  $\bigoplus h(M_i)$ . Set  $E^\vee(D) = End^\vee(H|_D)$  for any subgraph. When  $D$  is finite,  $E^\vee(D)$  is quotient of a finite sum of comodules of the form  $End(H(M)) \cong H(M)^{\dim H(M)}$ , so the claim follows when  $V = E^\vee(D)$ . In

general, the matrix coefficients of the  $E^\vee(\Delta)$  coaction on  $V$  lie in some  $E^\vee(D)$  with  $D$  finite. Thus  $V$  has a quotient of a finite sum of copies of  $E^\vee(D)$ . So the claim holds in general.  $\square$

**Remark 2.2.6.** *There is a dual description of  $End^\vee(H)$ -comod which is closer to what Nori originally used [L2, N2]. As in the previous argument, we can express  $E$ -comod  $= \cup E^\vee(D)$  as  $D \subset \Delta$  runs over finite subgraphs. Therefore as explained in appendix A, we have equivalences*

$$\begin{aligned} End^\vee(H)\text{-comod} &\sim \varinjlim_D E^\vee(D)\text{-comod} \\ &\sim \varinjlim_D End(H|_D)\text{-mod} \end{aligned}$$

**Theorem 2.2.7.** *If  $U : \mathcal{A} \rightarrow F\text{-mod}$  is an exact faithful  $F$ -linear functor on an  $F$ -linear abelian category, then  $\mathcal{A}$  is equivalent to  $End^\vee(U)$ -comod.*

*Proof.* The proof given in [JS, §7, thm 3] for the complex field works in general.  $\square$

It is instructive to observe that the corresponding statement for  $End(U)$ -mod will usually fail.

**Example 2.2.8.** *Let  $\mathcal{A}$  be the category of finite dimensional  $\mathbb{Z}$ -graded  $\mathbb{C}$ -vector spaces. This can be identified with the category of comodules over  $End^\vee(U) = \mathbb{C}[T, T^{-1}]$  in the usual way. However, the category of  $End(U) = \prod_{\mathbb{Z}} \mathbb{C}$  modules is much bigger. For example,  $\mathbb{C}$  with the  $End(U)$ -action arising from a nontrivial ultraproduct  $End(U) \rightarrow (\prod \mathbb{C}/U) \cong \mathbb{C}$  gives an  $End(U)$ -module which does not arise from a graded module.*

We can use this theorem to deduce a version of Nori's Tannakian theorem. (The original statement, which is stronger, can be found in [Br, thm 1], [L2, §3.3] or [N2].)

**Corollary 2.2.9** (Nori). *Suppose that  $\mathcal{A}$  is an  $F$ -linear abelian category equipped with a faithful exact functor  $U : \mathcal{A} \rightarrow F\text{-mod}$ . If  $G : \Delta \rightarrow \mathcal{A}$  is a morphism of directed graphs such that  $H$  is equivalent to  $U \circ G$ , then there is a functor  $\tilde{G} : End^\vee(H)\text{-comod} \rightarrow \mathcal{A}$  (called the extension of  $G$ ) rendering the diagram*

$$\begin{array}{ccc} \Delta & \xrightarrow{G} & \mathcal{A} \\ \downarrow h & \searrow H & \downarrow U \\ End^\vee(H)\text{-comod} & \xrightarrow{\tilde{G}} & F\text{-mod} \end{array}$$

*commutative up to natural equivalence.*

It is convenient to prove a slightly stronger statement, where  $F\text{-mod}$  is replaced by, for example, the category of finite dimensional modules over an  $F$ -algebra.

**Corollary 2.2.10.** *Let  $\mathcal{R}$  be an  $F$ -linear abelian category with a faithful exact functor  $\rho : \mathcal{R} \rightarrow F\text{-mod}$ . Suppose that  $H : \Delta \rightarrow F\text{-mod}$  factors as  $H_1 \circ \rho$  (up to natural equivalence). Suppose that  $\mathcal{A}$  is an  $F$ -linear abelian category equipped with a faithful exact functor  $U : \mathcal{A} \rightarrow \mathcal{R}$ . If  $G : \Delta \rightarrow \mathcal{A}$  is a morphism of directed graphs such that  $H_1$  is equivalent to  $U \circ G$ , then there are functors  $End^\vee(H)\text{-comod} \rightarrow \mathcal{R}$ ,  $\tilde{G} : End^\vee(H)\text{-comod} \rightarrow \mathcal{A}$  rendering the diagram*

$$\begin{array}{ccc}
 \Delta & \xrightarrow{G} & \mathcal{A} \\
 \downarrow h & \nearrow H_1 & \downarrow U \\
 End^\vee(H)\text{-comod} & \xrightarrow{\tilde{G}} & \mathcal{R} \\
 & \searrow U & \downarrow \rho \\
 & & F\text{-mod}
 \end{array}$$

commutative up to natural equivalence.

*Proof.* We obtain a commutative diagram of coalgebras

$$\begin{array}{ccc}
 End^\vee(H) & \longrightarrow & End^\vee(U \circ \rho) \\
 \downarrow & \swarrow & \\
 End^\vee(\rho) & & 
 \end{array}$$

Thus we get a functor between the categories of finite dimensional comodules

$$\begin{array}{ccc}
 End^\vee(H)\text{-comod} & \xrightarrow{\tilde{G}} & End^\vee(U \circ \rho)\text{-comod} \sim \mathcal{A} \\
 \downarrow & \swarrow & \\
 End^\vee(\rho)\text{-comod} & \sim & \mathcal{R}
 \end{array}$$

The equivalences of categories, indicated by  $\sim$ , follow from the theorem and the above assumptions.  $\square$

There is also a naturally statement, which we give only in the situation of corollary 2.2.9.

**Corollary 2.2.11.** *Given a diagram*

$$\begin{array}{ccc}
 \Delta & \xrightarrow{G} & \mathcal{A} \\
 \downarrow \pi & \nearrow H & \downarrow U \\
 \Delta' & \xrightarrow{G'} & \mathcal{A}' \\
 & \searrow H' & \downarrow U' \\
 & & F\text{-mod}
 \end{array}$$

which commutes up to natural isomorphism, the diagram

$$\begin{array}{ccc} \text{End}^\vee(H)\text{-comod} & \xrightarrow{\tilde{G}} & \mathcal{A} \\ \downarrow \pi & & \downarrow \\ \text{End}^\vee(H')\text{-comod} & \xrightarrow{\tilde{G}'} & \mathcal{A}' \end{array}$$

commutes up to natural isomorphism.

The case of particular interest to us in corollary 2.2.10 is the category  $\mathcal{R} = (F\text{-mod})^n$  of finite dimensional vector spaces admitting gradings of the form  $V = V_1 \oplus V_2 \oplus \dots \oplus V_n$ . This can be identified with the category of finite modules over the ring  $F^n$ . A natural example arises as follows. Given functors  $H_i : \Delta_i \rightarrow F\text{-mod}$ , we can define a new functor  $H_1 \times H_2 \times \dots \times H_n$  on the cartesian product  $\Delta_1 \times \Delta_2 \times \dots \times \Delta_n$  in the category of graphs, to  $(F\text{-mod})^n$  by

$$H_1 \times H_2 \times \dots \times H_n(M_1, \dots, M_n) = H_1(M_1) \oplus \dots \oplus H_n(M_n)$$

We have an induced functor

$$\text{End}^\vee(H_1 \times H_2 \times \dots \times H_n)\text{-comod} \rightarrow \prod_i (\text{End}^\vee(H_i)\text{-comod}) \quad (5)$$

where to be clear, on the left we really mean  $\text{End}^\vee(\rho \circ (H_1 \times \dots \times H_n))$ , where  $\rho : (F\text{-mod})^n \rightarrow F\text{-mod}$  is the forgetful functor. We will need a criterion for when this is an equivalence. It usually isn't.

**Example 2.2.12.** Let  $\Delta$  be a graph consisting of a single vertex  $pt$  and no morphisms. Let  $H(pt) = F$ . Since this is a finite graph, we can work with endomorphism rings rather than coalgebras. One has  $\text{End}(H) = F$  and  $\text{End}(H \times H) = M_2(F)$  the ring of  $2 \times 2$  matrices. Therefore by Morita's theorem,  $\text{End}(H \times H)\text{-mod} \sim F\text{-mod} \simeq (F\text{-mod})^2$ . The natural map  $\text{End}(H \times H)\text{-mod} \rightarrow (F\text{-mod})^2$  is the diagonal embedding  $F\text{-mod} \rightarrow (F\text{-mod})^2$ .

**Lemma 2.2.13.** Suppose that each object in  $\Delta_i$  has maps to an object  $\emptyset_i$  satisfying  $H_i(\emptyset_i) = 0$ . Then (5) is an equivalence.

*Proof.* It is enough to check this for  $n = 2$  graphs. By taking limits, we can reduce to the case where  $\Delta_i$  are both finite. From equation (1), the ring  $\text{End}(H_1 \times H_2)$  consists of families

$$(f_{P_1, P_2}) \in \prod \text{End}(H_1(P_1) \oplus H_2(P_2))$$

compatible with composition along morphisms. Choose maps  $\tau_i : P_i \rightarrow \emptyset_i$ . By considering compatibility along the morphisms  $1 \times \tau_2 : (P_1, P_2) \rightarrow (P_1, \emptyset_2)$  and  $\tau_1 \times 1 : (P_1, P_2) \rightarrow (\emptyset_1, P_2)$ , we see that  $f_{P_1, P_2}$  must be of the form

$$\begin{pmatrix} f_{P_1, \emptyset_2} & 0 \\ 0 & f_{\emptyset_1, P_2} \end{pmatrix}. \text{ Thus}$$

$$\text{End}(H_1 \times H_2) = \text{End}(H_1) \times \text{End}(H_2)$$

□

### 2.3 Enriched model

Let  $H : \Delta \rightarrow F\text{-mod}$  be a functor on a graph as above. Although functors on  $End^\vee(H)\text{-comod}$  can be constructed with the help of corollary 2.2.10, it is sometimes difficult to apply. It will be convenient to give an alternative description (up to equivalence) of  $End^\vee(H)\text{-comod}$  which allows us to incorporate extra structure. Fix a finite dimensional commutative  $F$ -algebra with an algebra homomorphism  $p : R \rightarrow F$  such that  $F$  is flat over  $R$ . These rather strong assumptions are valid in the case of principal interest to us, where  $R = F \times F$  with  $p$  projection onto the first factor. Suppose that  $H^\# : \Delta \rightarrow R\text{-mod}$  is a functor. We can define the  $R$ -coalgebra  $End_R^\vee(H^\#)$  by replacing  $Hom$  by  $Hom_R$  in (4). This is not the same as the coalgebra  $End^\vee(H^\# \circ \rho)$  considered earlier. Let  $End_R^\vee(H^\#)\text{-comod}$  denote the  $R$ -linear category of comodules in  $R\text{-mod}$ . Then we wish to describe the relationship between  $End^\vee(H^\#)\text{-comod}$  and  $End^\vee(H)\text{-comod}$ .

First, we make a brief digression. Given an  $F$ -linear abelian category  $C$ , an ideal  $I$  is a collection of subspaces  $I(c_1, c_2) \subseteq Hom(c_1, c_2)$  such that

$$Hom(c_2, c_3) \circ I(c_1, c_2) \subseteq I(c_1, c_3),$$

$$I(c_1, c_2) \circ Hom(c_0, c_1) \subseteq I(c_0, c_2)$$

Given an ideal  $I$ ,  $C/I$  is the category with the same objects as  $C$  and

$$Hom_{C/I}(c_1, c_2) = Hom_C(c_1, c_2)/I(c_1, c_2)$$

For example, if  $G : C \rightarrow D$  is an exact functor,  $\ker G = \{f \in Mor C \mid G(f) = 0\}$  is an ideal. Note, however, that the quotient  $C/\ker G$  should not be confused with the quotient of  $C$  by the thick subcategory generated by  $\{c \in Ob C \mid G(c) = 0\}$ .

**Lemma 2.3.1.** *If  $G : C \rightarrow D$  is an exact functor between essentially small abelian categories such that*

(a)  *$G$  is essentially surjective.*

(b)  *$G$  is surjective on Homs.*

*Then  $D$  is equivalent to  $C/\ker G$ .*

*Proof.*  $G$  induces an equivalence  $C/\ker G \sim D$  since  $Hom_C(c, c')/\ker G \cong Hom_D(G(c), G(c'))$   $\square$

Returning to the set up describe earlier. We have an isomorphism  $p : End^\vee(H^\#) \otimes_R F \cong End^\vee(H)$  and hence an exact functor

$$p : End^\vee(H^\#)\text{-comod} \rightarrow End^\vee(H)\text{-comod}$$

given by  $M \mapsto M \otimes_R F$ . The conditions of the above lemma are easily verified in general. In the case, where  $R = F^2$ , this is almost immediate.  $End^\vee(H^\#)\text{-comodule}$  decomposes into a sum of two factors corresponding to the idempotents of  $R$ , and  $p$  is projection on the first factor. Thus:

**Corollary 2.3.2.**  $End^\vee(H)$ -comod is equivalent to  $End^\vee(H^\#)$ -comod/ $\ker p$ .

**Corollary 2.3.3.** An  $F$ -linear functor on  $End_R^\vee(H^\#)$ -comod such that  $\ker(p)$  maps to zero, induces a functor on  $End^\vee(H)$ -comod.

## 2.4 Products

We need to incorporate tensor products into our story. The category of functors from graphs to  $F$ -mod forms a category with tensor product given as follows. Let  $H : \Delta \rightarrow F\text{-mod}$  and  $H' : \Delta' \rightarrow F\text{-mod}$  be two such functors. Then  $H \otimes H' : \Delta \times \Delta' \rightarrow F\text{-mod}$  is given by  $(M, N) \mapsto H(M) \otimes H'(N)$ . The one point graph  $\{*\}$  with  $* \mapsto F$  gives the unit making this into a tensor category, where for our purposes a tensor category over  $F$  is an  $F$ -linear additive category with a bilinear symmetric monoidal structure. We have

$$End^\vee(H \otimes H') \cong End^\vee(H) \otimes End^\vee(H')$$

[JS, §8, prop 1]. This yields a product

$$End^\vee(H)\text{-comod} \times End^\vee(H')\text{-comod} \rightarrow End^\vee(H \otimes H')\text{-comod}$$

When  $H = H'$  is equipped with a symmetric associative pairing  $H \otimes H \rightarrow H$  and a unit  $* \in Ob\Delta$ ,  $H(*) = F$ . Then  $End(H)$  becomes a commutative bialgebra. Thus  $End(H)$ -comod becomes a tensor category with a tensor preserving functor to  $F$ -mod given by the forgetful functor. With minor modifications to the proof of corollary 2.2.9, we have

**Corollary 2.4.1.** Suppose that  $H$  has a product as above. If in the hypothesis of corollary 2.2.10,  $\mathcal{R} = F\text{-mod}$ ,  $\mathcal{A}$  is an  $F$ -linear abelian tensor category, and the functors  $\rho, U, G$  are product preserving. Then  $\tilde{G}$  is also product preserving.

Recall [De3, sect 2] [L1, chap IV, sect 1] that a dual of an object  $M$  in a tensor category, with unit  $\mathbf{1}$ , is an object  $M^\vee$  equipped with morphisms  $\delta : \mathbf{1} \rightarrow M^\vee \otimes M$  and  $\epsilon : M \otimes M^\vee \rightarrow \mathbf{1}$  such that the compositions

$$M \xrightarrow{id \otimes \delta} M \otimes M^\vee \otimes M \xrightarrow{\epsilon \otimes id} M \quad (D1)$$

$$M^\vee \xrightarrow{\delta \otimes id} M^\vee \otimes M \otimes M^\vee \xrightarrow{id \otimes \epsilon} M^\vee \quad (D2)$$

yield the identities. Alternatively,  $M^\vee$  is characterized by the natural isomorphisms

$$Hom(X \otimes M, Y) \cong Hom(X, M^\vee \otimes Y) \quad (D3)$$

$$Hom(X \otimes M^\vee, Y) \cong Hom(X, M \otimes Y) \quad (D4)$$

In particular, the dual is unique up to isomorphism if it exists. A map  $f : M \rightarrow N$  yields a dual or transpose map  $f^\vee : N^\vee \rightarrow M^\vee$  if  $M, N$  both possess duals.



**Lemma 2.4.2.** *Given an exact sequence*

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

*if  $M_i^\vee$  exists for  $i = 1, 2$  then  $M_3^\vee$  exists.*

*Proof.* Set  $M_3^\vee = \ker(M_2^\vee \rightarrow M_1^\vee)$ . Condition (D3) is a consequence of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(X \otimes M_3, Y) & \longrightarrow & \text{Hom}(X \otimes M_2, Y) & \longrightarrow & \text{Hom}(X \otimes M_1, Y) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Hom}(X, M_3^\vee \otimes Y) & \longrightarrow & \text{Hom}(X, M_2^\vee \otimes Y) & \longrightarrow & \text{Hom}(X, M_1^\vee \otimes Y) \end{array}$$

and (D4) is similar.  $\square$

A neutral Tannakian category over  $F$  is an abelian tensor category over  $F$ , with a faithful exact tensor preserving functor to  $F\text{-mod}$ , such that every object possesses a dual. Such a category can be realized as the category of comodules over a commutative Hopf algebra.

**Proposition 2.4.3.** *Suppose that  $H : \Delta \rightarrow F\text{-mod}$  is equipped with a symmetric associative product as above. Assume that for every object  $M \in \text{Ob}\Delta$ ,  $h(M)$  has a dual. Then  $\text{End}^\vee(H)\text{-comod}$  is neutral Tannakian.*

*Proof.* The proposition follows from lemmas 2.2.5 and 2.4.2  $\square$

## 3 Premotivic sheaves

### 3.1 Constructible sheaves

We recall:

**Definition 3.1.1.** *If  $X$  is a complex variety (defined over  $k \subset \mathbb{C}$ ), a sheaf  $\mathcal{F}$  on  $X_{an}$  is called constructible (or  $k$ -constructible) if it has finite dimensional stalks and there exists a partition  $\Sigma$  of  $X$  into Zariski locally closed (defined over  $k$ ) so that  $\mathcal{F}|_\sigma$  are locally constant for each  $\sigma \in \Sigma$ . In this case,  $\mathcal{F}$  is also called  $\Sigma$ -constructible. Let  $\text{Cons}(X)$  or  $\text{Cons}(X, \Sigma)$  denote the category of these.*

The definition of constructibility for sheaves on the étale topology  $X_{et}$  is similar [Mi, chap V], [SGA4, exp IX]. Basic examples of constructible sheaves include the direct images  $R^i f_* \mathcal{F}$  and more generally direct images of constructible sheaves [V1, cor. 2.4.2]. We give a slight refinement below (theorem 3.1.10).

**Definition 3.1.2.** *Given a morphism  $f : X \rightarrow S$  and a sheaf  $\mathcal{F}$  on  $X_{an}$  or  $X_{et}$ , we say that  $H_S^i(X, \mathcal{F}) = R^i f_* \mathcal{F}$  commutes with base change if for any quasi-projective morphism  $g : T \rightarrow S$  the canonical base change map*

$$g^* H_S^i(\mathcal{F}) \rightarrow H_T^i(X \times_S T, g^* \mathcal{F})$$

*is an isomorphism.*

**Definition 3.1.3.** Given a morphism  $f : X \rightarrow S$  and a sheaf  $\mathcal{F}$  on  $X$ , if  $H_S^i(X, \mathcal{F})$  commutes with base change for all  $i$ , we will say that  $\mathcal{F}$  has the base change property (with respect to  $f$ ).

The condition implies that

$$H_S^i(X, \mathcal{F})_s \rightarrow H^i(X_s, \mathcal{F}|_{X_s})$$

is an isomorphism for every  $s \in S$ . Open immersions  $X \rightarrow S$  give examples where this will fail for  $s \in S - X$ . We review some (known) criteria for this to hold. A morphism  $f : X \rightarrow S$  will be called *locally trivial* if it is a topological (although not necessarily an analytic) locally trivial fibre bundle with respect to the the analytic topology. More generally:

**Definition 3.1.4.** Let say that the pair  $(f : X \rightarrow S, \mathcal{F} \in \text{Cons}(X))$  is locally trivial if there exists an open cover  $\{U_i\}$  of  $S$  and a stratified space  $\Phi$  with a constructible sheaf  $\mathcal{G}$ , such that there are stratified homeomorphisms  $f^{-1}U_i \cong \Phi \times U_i$  compatible with  $f$  such that  $\mathcal{G}$  pulls back to the restriction of  $\mathcal{F}$ .

**Theorem 3.1.5.**

1. Given a short exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

if any two of the  $\mathcal{F}_i$  have the base change property, then so does the third.

2. If  $(f : X \rightarrow S, \mathcal{F})$  is locally trivial, then  $\mathcal{F}$  has the base change property.
3. (Proper base change) If  $f : X \rightarrow S$  is proper, then any sheaf has the base change property.
4. (Locally trivial base change) If  $T \rightarrow S$  is locally trivial, then the base map for  $f : X \rightarrow S$  with respect to  $T$  is an isomorphism for any  $\mathcal{F}$  and  $i$ .

*Proof.* The first statement is obvious. The second is clear once we observe that it can be reduced to the case of a product  $S \times \Phi \rightarrow S$ , with  $\mathcal{F}$  pulled back from  $\Phi$ . For the third, when  $f : X \rightarrow S$  is proper and  $T$  is a point, the base change property follows from [L, thm 6.2]. Therefore

$$g^* H_S^i(\mathcal{F}) \rightarrow H_T^i(X \times_S T, g^* \mathcal{F})$$

is an isomorphism on stalks.

For the fourth statement, we can reduce to the case of product, and then apply the Künneth formula.  $\square$

We can combine these criteria into one convenient notion.

**Definition 3.1.6.** Given a quasi-projective morphism  $f : X \rightarrow S$  and a sheaf  $\mathcal{F} \in \text{Cons}(X)$ , we will say that the pair  $(f, \mathcal{F})$  is controlled, or that  $f$  is controlled with respect to  $\mathcal{F}$ , if there exists a commutative diagram

$$\begin{array}{ccccc}
 & & & & S \\
 & & & & \downarrow = \\
 & & & & S \\
 X & \xrightarrow{g} & T & \xrightarrow{h} & S \\
 \downarrow j & \nearrow \bar{g} & & & \\
 \bar{X} & & & & 
 \end{array}$$

such that  $h$  and  $\bar{g}$  are projective,  $j$  is an open immersion, and such that  $(\bar{X}, j_! \mathcal{F})$  is locally trivial over  $T$ .

It is worth observing that the condition is automatic if  $S$  is point because everything is locally trivial over a point. Also note that in general the conditions imply that  $(\bar{X}, \bar{X} - X)$  is a relative fibre bundle over  $T$ . Such a diagram, which need not be unique, will be called a control diagram for the pair  $(f, \mathcal{F})$ .

**Lemma 3.1.7.** If  $(f : X \rightarrow S, \mathcal{F})$  is controlled, then  $\mathcal{F}$  has the base change property with respect to  $f$ .

*Proof.* Choose a control diagram as above. Let  $q : S' \rightarrow S$  be a morphism, and consider the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{g} & T & \xrightarrow{h} & S \\
 \uparrow q_2 & & \uparrow q_1 & & \uparrow q \\
 X' & \xrightarrow{g'} & T' & \xrightarrow{h'} & S'
 \end{array}$$

where the squares are Cartesian. We have to prove that  $q^* R^i(h \circ g)_* \mathcal{F} \cong R^i(h \circ g')_* \mathcal{F}$ . It is enough to check isomorphism on the  $E_2$  terms of Leray spectral sequence. We have

$$q^* R^a h_* R^b g_* \mathcal{F} \cong R^a h'_* q_1^* R^b g'_* \mathcal{F}$$

because  $h$  is proper, and we have

$$R^a h'_* q_1^* R^b g'_* \mathcal{F} \cong R^a h_* R^b g_* \mathcal{F}$$

because  $g$  is locally trivial with respect to  $\mathcal{F}$ . □

**Proposition 3.1.8.** Suppose that  $(f : X \rightarrow S, \mathcal{F})$  is a controlled pair with a locally closed  $S$ -embedding  $X \rightarrow \mathbb{P}^N \times S$ . Given a nonempty Zariski open set  $P \subset \mathbb{P}^N$ , there exists a Zariski open cover  $\{S_\alpha\}$  of  $S$  and elements  $H_\alpha \in P$ , such that

$$\begin{aligned}
 (H_\alpha \cap f^{-1} S_\alpha \rightarrow S_\alpha, \mathcal{F}|_{H_\alpha}) \\
 (f^{-1} S_\alpha \rightarrow S_\alpha, q_{\alpha!} q_\alpha^* \mathcal{F})
 \end{aligned}$$

are controlled, where  $q_\alpha : f^{-1} S_\alpha - H_\alpha \rightarrow f^{-1} S_\alpha$  is the inclusion.

*Proof.* Choose a control diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & T & \xrightarrow{h} & S \\ \downarrow j & \nearrow \bar{g} & & & \\ \bar{X} & & & & \end{array}$$

for  $\mathcal{F}$ . Then by assumption,  $j_!\mathcal{F}$  is constructible with respect to a stratification  $\{\bar{X}_\bullet\}$  of  $\bar{X}$  which is locally trivial over  $S$ . Given  $s \in S$ , we may choose  $H \in P$  transverse to  $\bar{X}_\bullet \cap \bar{X}_s$ . It remains transverse to  $\bar{X}_\bullet \cap \bar{X}_t$ , for  $t$  in a neighbourhood  $S_s$  of  $s$ . It follows that the stratification generated by  $\bar{X}_\bullet$  and  $H$  and  $\bar{X}_\bullet \cap H$  are locally trivial over  $S_s$ .  $\square$

In order to proceed, we will need Whitney stratifications. For our purposes a stratification of a variety  $X$  is a finite partition  $\Sigma$  of  $X$  into Zariski locally closed sets such that the closure of any stratum  $\sigma \in \Sigma$  is a union of strata. When  $X$  is complex, we will say that this is Whitney if any stratum is smooth and if the Whitney conditions hold for any  $x \in \sigma \subset \bar{\tau}$  [Lil, Mr, T, V2]. This means that given sequences  $x_i \in \sigma$  and  $y_i \in \tau$ , both converging to  $x$ , the limit of the secants  $\overline{x_i y_i}$  (with respect to a local embedding into  $\mathbb{C}^N$ ) lies in the limit of tangent spaces  $T_{\tau, y_i}$  when the limits exist. These conditions may appear strange at first glance, but their importance comes from the fact that they imply local triviality of the topology of  $X$  along each stratum  $\sigma$ . In more precise terms, there exists a tubular neighbourhood  $\sigma \subset T_\sigma \subset X^{an}$  [Mr] with a retraction  $\pi : T_\sigma \rightarrow \sigma$  which makes it a locally trivial fibre bundle with a contractible fibre.

A number of authors have observed that the Whitney conditions can be reformulated in more algebraic language; a simple description can be found in the proof of [Lil, thm 3.2] for instance. So in particular, given  $k$ -variety  $X$ , a stratification which is Whitney for one embedding  $k \subset \mathbb{C}$  will be Whitney for all. Concerning existence in this generality, we have

**Lemma 3.1.9.** *If  $X$  is a  $k$ -variety with a filtration  $X = Y^0 \supset Y^1 \supset \dots \supset Y^n = \emptyset$  by closed sets, there exists a Whitney stratification defined over  $k$  such that each  $Y^i$  is a union of strata.*

*Sketch.* This is a modification of Teissier's method for constructing canonical Whitney stratifications [T]. For simplicity, assume that  $X$  is irreducible. Define  $X^0 = X$ ,  $X^1 = X_{sing} \cup Y^1$  and inductively set

$$X^{i+1} = \{x \in X^i \mid \exists j < i, \text{ the Whitney conditions fail at } x \in X^i \subset X^j\} \cup Y^{i+1}$$

This gives a chain of closed sets decreasing to  $\emptyset$  by [T, p 477]. The partition  $\{X^i - X^{i+1}\}$  can be seen to give a Whitney stratification by arguing as in [T, pp 478-480].  $\square$

The following gives a version of Deligne's generic base change theorem [SGA4h, thm 1.9, p 240] for complex varieties.

**Theorem 3.1.10.** *Given a morphism  $f : X \rightarrow Y$  defined over  $k$  and a  $k$ -constructible sheaf  $\mathcal{F}$  on  $X$ , the sheaves  $R^i f_* \mathcal{F}$  are  $k$ -constructible. There exists a dense Zariski open  $U \subset S$  such that the restriction  $\mathcal{F}$  has the base change property with respect to  $f^{-1}U \rightarrow U$ .*

*Proof.* Let  $j : X \hookrightarrow \bar{X}$  be an open immersion such that there is a proper map  $\bar{f} : \bar{X} \rightarrow Y$  extending  $f$ . Let  $\Sigma$  be a Whitney stratification of  $Y$  with connected strata, and let  $\Lambda$  be a Whitney stratification of  $X$  refining  $f^{-1}\Lambda$  such that  $j_* \mathcal{F}$  is  $\Sigma$ -constructible. By lemma 3.1.9, we may assume that  $\Sigma$  and  $\Lambda$  are defined over  $k$ . We may also assume that  $\bar{X} - X$  is a union of strata, and that  $\bar{f}$  is a submersion on each stratum. Each  $\sigma \in \Sigma$  possesses a tubular neighbourhood  $\sigma \subset T_\sigma \subset Y$  with a retraction  $\pi : T_\sigma \rightarrow \sigma$  which makes it a locally trivial fibre bundle with a contractible fibre  $G$ . The preimage  $f^{-1}T_\sigma$  inherits a stratification from  $X$ , such that  $f^{-1}\sigma$  is a union of strata. Thom's isotopy theorem [Mr, V2] implies that  $f^{-1}T_\sigma \rightarrow T_\sigma$  is a stratified fibre bundle. That is, there exists an open cover  $\{V_i\}$  of  $T_\sigma$  and stratified space  $\Phi$  such that there are homeomorphisms  $f^{-1}V_i \cong \Phi \times V_i$  of stratified spaces compatible with projection. One can see that there is no loss in generality in assuming that each  $V_i = \pi^{-1}U_i$  for an open subset  $U_i \subset \sigma$ . We may assume that the  $U_i$  are contractible. It follows that  $(f^{-1}T_\sigma, f^{-1}\sigma)$  is a (relative) fibre bundle over  $\sigma$  with fibre say  $(\Phi \times G, \Phi')$ . We can see that  $\Phi$  carries a constructible sheaf  $\mathcal{G}$  which pulls back to the restriction of  $\mathcal{F}$  under the homeomorphisms  $f^{-1}U_i \cong \Phi \times G \times U_i$ . This implies that  $R^i f_* \mathcal{F}$  is locally constant along  $\sigma$ , and hence  $k$ -constructible.

Applying the above argument to a Zariski dense stratum  $\sigma$ , shows that  $(X \rightarrow Y, \mathcal{F})$  is a locally trivial over  $\sigma$ . Therefore the base change property holds over  $\sigma$ .  $\square$

## 3.2 Cohomology of pairs

Let  $S$  be a  $k$ -variety. Let  $Var_S^2$  be the category whose objects are pairs  $(X \rightarrow S, Y)$  with  $Y \subseteq X$  closed. A morphism from  $(X \rightarrow S, Y) \rightarrow (X' \rightarrow S, Y')$  is a morphism of  $S$ -schemes  $X \rightarrow X'$  such that  $f(Y) \subseteq Y'$ . For such an object and a sheaf  $\mathcal{F}$  on  $X_{an}$ , set

$$H_S^i(X, Y; \mathcal{F}) = R^i f_* j_{X, Y!} \mathcal{F}|_{X-Y}$$

where  $f : X \rightarrow S$  is the projection and  $j_{X, Y} : X - Y \rightarrow X$  is the inclusion. We revert to writing this as  $H_S^i(X, \mathcal{F})$  when  $Y$  is empty. When  $\mathcal{F} = F$  is constant,  $H_S^i(X, Y; F)$  is  $k$ -constructible by the theorem 3.1.10, and we can describe this as the sheaf associated to

$$U \mapsto H^i(f^{-1}U, f^{-1}U \cap Y; F)$$

The map  $(X, Y) \mapsto H_S^i(X, Y; F)$  is easily seen to give a contravariant functor on  $Var_S^2$ . The morphisms  $H_S^i(X, Y) \rightarrow H_S^i(X', Y')$  are induced by the homomorphisms

$$H^i(f'^{-1}U, f'^{-1}U \cap Y'; F) \rightarrow H^i(f^{-1}U, f^{-1}U \cap Y; F)$$

**Definition 3.2.1.** A pair  $(f : X \rightarrow S, Y)$  in  $\text{Var}_S^2$  is controlled with respect to  $\mathcal{F}$  if  $f$  is controlled with respect to the sheaf  $j_{X,Y!}\mathcal{F}|_{X-Y}$ . The pair is said to be controlled if it so with respect to the constant sheaf  $F$ .

The control condition for a pair, with respect to  $F$ , amounts to requiring that both  $(\bar{X}, \bar{X} - X)$  and  $(\bar{Y}, \bar{Y} - Y)$  are relative fibre bundles over an intermediate projective family  $T \rightarrow S$ , where  $\bar{Y}$  is the closure of  $Y$ .

**Lemma 3.2.2.** If  $f : (X \rightarrow S, Y)$  is controlled with respect to  $\mathcal{F}$ , then  $j_{X,Y!}\mathcal{F}|_{X-Y}$  has the base change property with respect to  $f$ .

*Proof.* This is a consequence of lemma 3.1.7. □

Therefore if  $(X \rightarrow S, Y)$  is controlled then

$$H_S^i(X, Y; F)_s \cong H^i(X_s, Y_s; F) \quad (6)$$

for every  $s \in S$ .

From proposition 3.1.8, we obtain.

**Lemma 3.2.3.** Suppose that  $(X \rightarrow S, Y)$  is controlled. Then for a general hyperplane  $H$  (with respect to a locally closed embedding  $X \subset \mathbb{P}^N \times S$ ),  $(H \rightarrow S, H \cap Y)$  is controlled.

Given a chain of closed sets  $X \supset Y \supset Z$  and a sheaf  $\mathcal{F}$  on  $X$ , we get an exact sequence

$$0 \rightarrow j_{X,Y!}\mathcal{F}|_{X-Y} \rightarrow j_{X,Z!}\mathcal{F}|_{X-Z} \rightarrow i_*j_{Y,Z!}\mathcal{F}|_{Y-Z} \rightarrow 0$$

where  $i : Y \rightarrow X$  is the inclusion. This induces a long exact sequence

$$\dots H_S^i(X, Y; \mathcal{F}) \rightarrow H_S^i(X, Z; \mathcal{F}) \rightarrow H_S^i(Y, Z; \mathcal{F}) \rightarrow H_S^{i+1}(X, Y; \mathcal{F}) \dots$$

which reduces to the usual exact sequence for pairs, when  $S$  is point and  $\mathcal{F}$  is constant.

### 3.3 Premotivic sheaves

Let  $S$  be a  $k$ -variety. The category  $\mathcal{PM}(S)$  of premotivic sheaves is constructed as a direct limit of categories  $\mathcal{PM}(S, \Sigma)$ . Each  $\mathcal{PM}(S, \Sigma)$  is obtained by applying Nori's construction to an appropriate graph  $\Delta(S, \Sigma)$  and functor  $H_\Sigma$  given below.

Let  $S \in \text{ObVar}_k$  be *connected*. Then we construct a graph  $\Delta(S)$  as follows. The objects (i.e. vertices) are quadruples  $(X \rightarrow S, Y, i, w)$  consisting of

- (1) a quasi-projective morphism  $X \rightarrow S$ .
- (2) a closed subvariety  $Y$  such that the pair  $(X \rightarrow S, Y)$  is controlled (definition 3.2.1),
- (3) a natural number  $i \in \mathbb{N}$  and an integer  $w$ .

The set of morphisms (edges) is the union of the three following sets:

Type I: Geometric morphisms

$$(X \rightarrow S, Y, i, w) \rightarrow (X' \rightarrow S, Y', i, w)$$

where  $(X \rightarrow S, Y) \rightarrow (X' \rightarrow S, Y')$  is a morphism in  $Var_S^2$ .

Type II: Connecting morphisms

$$(f : X \rightarrow S, Y, i + 1, w) \rightarrow (f|_Y : Y \rightarrow S, Z, i, w)$$

for every chain  $Z \subseteq Y \subseteq X$  of closed sets.

Type III: Twisted projection morphisms

$$(X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i + 2, w + 1) \rightarrow (X, Y, i, w)$$

for every  $(X, Y, i, w) \in Ob\Gamma(S)$ .

For arbitrary  $S \in Var_k$ , set  $\Delta(S) = \coprod \Delta(S_i)$ , where  $S_i$  are the connected components. Thus the parameters  $i$  and  $w$  are locally constant.

By a good stratification or simply stratification of  $S$ , we mean a finite partition  $\Sigma$  into connected locally closed sets (defined over  $k$ ) such that  $\Sigma$  contains the closure of every element. Given a stratification  $\Sigma$ , let  $\Delta(S, \Sigma) \subset \Delta(S)$  be the full subgraph consisting of objects such that  $H_S^i(X, Y; F)$  is constructible with respect to the stratification  $\Sigma$ . We have that  $\bigcup \Delta(S, \Sigma) = \Delta(S)$  because the sheaves  $H_S^i(X, Y; F)$  are constructible.

Given  $\Sigma$  as above, let  $s = (s_\sigma \in \sigma(\bar{k}))$  denote a collection of base points, one for each  $\sigma \in \Sigma$ . Let  $|\Sigma|$  be the cardinality of  $\Sigma$ . Define

$$H_{\Sigma, s, F}(X, Y, i, w) = \prod_{\sigma \in \Sigma} [H_S^i(X, Y; F)]_{s_\sigma} = \prod_{\sigma \in \Sigma} H^i(X_{s_\sigma}, Y_{s_\sigma}; F)$$

to be the product of stalks. We usually suppress the symbols  $\Sigma, s, F$ .

We want to extend  $H = H_{\Sigma, s, F}$  to a functor  $\Delta(S, \Sigma)^{op} \rightarrow F\text{-mod}$ . We do this case by case.

Type I: A morphism  $g : (f : X \rightarrow S, Y, i, w) \rightarrow (f' : X' \rightarrow S, Y', i, w)$  of type I gives rise to the natural homomorphism

$$H^i(f'^{-1}U, f'^{-1}U \cap Y'; F) \rightarrow H^i(f^{-1}U, f^{-1}U \cap Y; F)$$

for each  $U \subset S$ . Since this is clearly a morphism of presheaves, it induces a morphism of sheaves  $H_S^i(X', Y') \rightarrow H_S^i(X, Y)$ . Thus we get the desired map  $H(X', Y', i, w) \rightarrow H(X, Y, i, w)$  by taking the product of this sheaf map over stalks. We give a second description which is a bit more

complicated, although better for comparing to the étale case. We have a commutative diagram

$$\begin{array}{ccccc}
j_{X'Y'}!F_{X'-Y'} & \xleftarrow{[1]} & & & F_{Y'} \\
\downarrow \text{dotted} & \searrow & & & \downarrow \\
& & F_{X'} & & \\
& & \downarrow & & \\
\mathbb{R}g_*j_{XY}!F_{X-Y} & \xleftarrow{[1]} & & & \mathbb{R}g_*g^*F_{Y'} \\
& \searrow & & & \downarrow \\
& & \mathbb{R}g_*g^*F_{X'} & & 
\end{array}$$

where the triangles are distinguished, and the solid vertical arrows are the adjunction homomorphisms. Thus we get the dotted arrow above. From which we obtain

$$\mathbb{R}f'_*j_{X'Y'}!F \rightarrow \mathbb{R}f'_*\mathbb{R}g_*j_{XY}!F \cong \mathbb{R}f_*j_{XY}!F$$

So we get a map of sheaves

$$R^i f'_*j_{X'Y'}!F \rightarrow R^i f_*j_{XY}!F$$

which is easily seen to coincide with the previous map.

Type II: A morphism  $(X, Y, i+1, w) \rightarrow (Y, Z, i, w)$  of type II gives rise to a connecting homomorphism  $H_S^i(Y, Z) \rightarrow H_S^{i+1}(X, Y)$  induced from the exact sequence

$$0 \rightarrow j_{X,Y}!F \rightarrow j_{X,Z}!F \rightarrow j_{Y,Z}!F \rightarrow 0$$

Taking a product over stalks yields  $H(Y, Z, i, w) \rightarrow H(X, Y, i+1, w)$ .

Type III: Finally a morphism  $(X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i+2, w+1) \rightarrow (X, Y, i, w)$  corresponds to the isomorphism

$$H_S^i(X, Y; F) \rightarrow H_S^{i+2}(X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}; F)$$

given by exterior product with the fundamental cycle of  $(\mathbb{P}^1, \{0\})$ . This gives rise to

$$H(X, Y, i, w) \rightarrow H(X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i+2, w+1)$$

Thus we can apply the construction from the previous section to obtain:

**Definition 3.3.1.** *The category  $\mathcal{PM}(S, \Sigma, s; F)$  of  $\Sigma$ -constructible premotivic sheaves of  $F$ -modules on  $S$  is the category of finite dimensional right comodules over  $\text{End}^\vee(H_{\Sigma, s})$ . For any finite commutative  $F$ -algebra  $R$ , let  $\mathcal{PM}(S, \Sigma, s; F) \otimes_F R$  denote the category with finitely generated right comodules over  $\text{End}^\vee(H) \otimes_F R$ .*



### 3.4 Realizations

By definition there is a faithful exact forgetful functor  $U : \mathcal{PM}(S, \Sigma; F) \rightarrow F\text{-mod}$ . We can see immediately from the universal coefficient theorem that  $\mathcal{PM}(S, R) = \mathcal{PM}(S, F) \otimes_F R$ , whenever  $F \subseteq R$  is a field extension. The matrix coefficients of the  $End^\vee(H)$ -coaction of any object  $V$  of  $\mathcal{PM}(S, \Sigma)$  lie in some  $End^\vee(H|_D)$  for a finite subgraph  $D$ . Thus  $V$  can be regarded as an  $End^\vee(H|_D)$ -comodule, or equivalently an  $End(H|_D)$ -module. In fact, we can describe  $\mathcal{PM}(S, \Sigma, s)$  as the direct limit of the categories of finite dimensional  $End(H|_D)$ -modules, as  $D \subset \Delta(S)$  varies over finite subgraphs (cf [Br]). This dual description was employed by Nori in his work, and it would appear that  $\mathcal{PM}(Spec k, Spec \mathbb{C})$  is just Nori's category of cohomological motives tensored with  $F$ . We write this as  $\mathcal{PM}(k; F)$  or simply  $\mathcal{PM}(k)$  from now on.

Given  $M = (X \rightarrow S, Y, i, w) \in Ob\Delta(S)$ ,  $H(M)$  is naturally an  $End(H(M))$ -module, and hence by transpose an  $End^\vee(H)$ -comodule denoted by  $h_S^i(X, Y)(w)$  or  $h_S^i(X, Y)$  if  $w = 0$  (we will see shortly that this independent of  $\Sigma$  and  $s$  in a suitable sense). When  $S = Spec k$ , we omit the subscript.

By definition we have

**Proposition 3.4.1.**  *$\mathcal{PM}(S, \Sigma, s; F)$  is an  $F$ -linear abelian category with an exact faithful functor to  $F\text{-mod}$ .*

In view of the following, we may suppress base points.

**Lemma 3.4.2.** *Suppose that  $t_\sigma$  is another collection of base points, then  $\mathcal{PM}(S, \Sigma, s)$  and  $\mathcal{PM}(S, \Sigma, t)$  are isomorphic.*

*Proof.* Given a homotopy class of paths  $\gamma_\sigma$  in  $\sigma$  joining  $s_\sigma$  to  $t_\sigma$ , parallel transport along these curves yields an isomorphism of fiber functors  $H_s \cong H_t$ .  $\square$

**Remark 3.4.3.** *This business of choosing base points and then suppressing them is a bit clumsy. A more elegant approach is to simply redefine*

$$H_{\Sigma, F}(X, Y, i, w) = \prod_{\sigma \in \Sigma} \Gamma(\tilde{\sigma}, \pi_\sigma^* H_S^i(X, Y; F))$$

where  $\pi_\sigma : \tilde{\sigma} \rightarrow \sigma(\mathbb{C})$  are the universal covers, and then build  $\mathcal{PM}(S, \Sigma; F)$  accordingly. However, the original approach does make certain things more transparent, and will generally be preferred.

We have the following consequences of corollary 2.2.10.

**Construction 3.4.4.** *Let  $Cons(S_{l, an}, \Sigma; F)$  denote the category of sheaves of  $F$ -modules which are constructible with respect to the stratification  $\Sigma$ . The fibre functor  $\Phi : Cons(S_{an}, \Sigma) \rightarrow F\text{-mod}$  given the product of stalks at the base points provides a faithful exact functor. The discussion from the previous section shows that  $(X, Y, i, w) \mapsto H_S^i(X, Y; F)$  is a functor on  $\Delta(S, \Sigma)^{op}$  and that  $H$  is a composition of this with  $\Phi$ . Thus corollary 2.2.10 yields an extension functor  $R_{l, B} = R_B : \mathcal{PM}(S, \Sigma) \rightarrow Cons(S_{l, an}, \Sigma)$  that we call Betti realization.  $R_B$  coincides with the forgetful functor  $U$  on  $\mathcal{PM}(k)$ .*

**Construction 3.4.5.** *The map*

$$t^n(X, Y, i, w) = h_S^i(X, Y)(w + n)$$

extends to a functor  $\Delta(S, \Sigma)^{op} \rightarrow \mathcal{PM}(S, \Sigma)$  satisfying  $t^n t^m = t^{n+m}$ . When composed with the forgetful functor to  $F\text{-mod}$ , we obtain  $H$ . Thus this extends to an endofunctor  $T^n : \mathcal{PM}(S, \Sigma) \rightarrow \mathcal{PM}(S, \Sigma)$  satisfying

$$T^n(h_S^i(X, Y)(w)) = h_S^i(X, Y)(w + n)$$

and  $T^n T^m = T^{n+m}$ ; in particular, it is an automorphism.

**Construction 3.4.6.** *Let  $F$  be finite or  $\mathbb{Q}_\ell$ . Let  $\bar{k} \subseteq \mathbb{C}$  denote the algebraic closure of  $k$ . Consider the map*

$$(f : X \rightarrow S, Y, i, w) \mapsto R^i \bar{f}_* j_{\bar{X}, \bar{Y}}^{et} F_{\bar{X} - \bar{Y}}(w)$$

where the sheaves and operations are on the étale topology,  $\bar{f} : \bar{X} \rightarrow S$  etc. are the base changes to  $\bar{k}$ ,  $j_{\bar{X}, \bar{Y}} : \bar{X} - \bar{Y} \rightarrow \bar{X}$  is the inclusion, and  $(w)$  represents the Tate twist. This is easily seen to be a functor by modifying the above discussion. Thanks to the comparison theorem between étale and classical cohomology (appendix B).

$$(R^i \bar{f}_* j_{\bar{X}, \bar{Y}}^{et} F_{\bar{X} - \bar{Y}}(w))_s \cong (R^i \bar{f}_* j_{X, Y} F_{X - Y})_s \text{ in } F\text{-mod}$$

for  $s \in S(\mathbb{C})$ . Thus we can obtain an extension which is the étale realization functor  $R_{et}$  from  $\mathcal{PM}(S; F)$  to the category  $\text{Cons}(S_{et}, \Sigma, ; F)$  of sheaves of  $F$ -modules on  $S_{et}$  constructible with respect to  $\Sigma$ .

**Construction 3.4.7.** *Let  $\text{Cons-MHM}(S_{l, an}, \Sigma; \mathbb{Q})$  denote the heart of the classical  $t$ -structure on the category  $\text{MHM}(S, \Sigma)$  of  $\Sigma$ -constructible mixed Hodge modules (appendix C). We have an embedding*

$$\text{rat} : \text{Cons-MHM}(S_{an}, \Sigma; \mathbb{Q}) \hookrightarrow \text{Cons}(S_{an}, \Sigma)$$

which can be composed with the above functor  $\Phi$  to obtain a fibre functor. Consider the map

$$(f : X \rightarrow S, Y, i, w) \mapsto {}^c H^i \circ \mathbb{R} \bar{f}_* j_{\bar{X}, \bar{Y}} F_{\bar{X} - \bar{Y}}(w)$$

where the operations are in the derived categories of mixed Hodge modules, and  ${}^c H^i = {}^c \tau_{\leq i} {}^c \tau_{\geq i}$  is cohomology with respect to the classical  $t$ -structure. When composed with  $\text{rat}$ , we obtain  $H_S^i(X, Y)$ . Thus we obtain a Hodge realization functor

$$R_{\iota, H} = R_H : \mathcal{PM}(S, \Sigma) \rightarrow \text{Cons-MHM}(S_{l, an}, \Sigma)$$

A special given in section 6.1 can be made more explicit.

We fixed an embedding  $\iota : k \hookrightarrow \mathbb{C}$  at the outset. Let write  $\mathcal{PM}(S; F)^\iota$  for the resulting category. We now show that the category  $\mathcal{PM}(S; F)$  is independent of this.

**Proposition 3.4.8.** *For any two embeddings of  $\iota, \mu : k \subset \mathbb{C}$ , the categories  $\mathcal{PM}(S, F)^\iota$  and  $\mathcal{PM}(S, F)^\mu$  are equivalent.*

*Proof.* It suffices to show that  $\mathcal{PM}(S, \Sigma, F)^\iota$  and  $\mathcal{PM}(S, \Sigma, F)^\mu$  are equivalent for every stratification. We note that  $\mathcal{PM}(S, \Sigma, ; F) = \mathcal{PM}(S, \Sigma; F_0) \otimes_{F_0} F$  for any subfield. Thus it suffices to assume that  $F$  is the prime field. Suppose that  $F = \mathbb{Z}/p\mathbb{Z}$ . Then, we can see this immediately by the comparison theorem [SGA4, exp XVI, thm 4.1]

$$\begin{aligned} H_{\Sigma, s}(X, Y, i, w; F) &= \prod_{\sigma \in \Sigma} [R^i f_* j_! F]_{s_\sigma} \\ &\cong \prod_{\sigma \in \Sigma} [R^i \bar{f}_* j_!^{et} F]_{s_\sigma} \\ &= H_{\Sigma, s}^{et}(X, Y, i, w; F) \end{aligned}$$

This description is independent of the embedding. Therefore  $\mathcal{PM}(S, \Sigma, F)^\iota$  and  $\mathcal{PM}(S, \Sigma, F)^\mu$  are equivalent.

The remaining case  $F = \mathbb{Q}$  follows Nori's argument [N2] quite closely. Write  $H^\iota$  and  $H^\mu$  for the functors corresponding to the embeddings. Define the category  $\mathcal{T}$  of triples  $(A, B, h)$   $A, B \in \text{Ob } F\text{-mod}$ ,  $h : A \otimes \mathbb{Q}_\ell \cong B \otimes \mathbb{Q}_\ell$ , where morphisms are compatible pairs of linear maps. If  $p$  denote the first projection  $(A, B, h) \mapsto A$ , then  $p$  is easily seen to be fully faithful and essentially surjective. Therefore it is an equivalence. So there is functor  $q : \mathbb{Q}\text{-mod} \rightarrow \mathcal{T}$  and natural isomorphisms  $\gamma : q \circ p \cong 1_{\mathcal{T}}$  and  $\eta : p \circ q \cong 1_{\mathbb{Q}\text{-mod}}$ . We get a functor  $H^{\mathcal{T}} : \Delta(S, \Sigma) \rightarrow \mathcal{T}$  by taking  $H^\iota$  and  $H^\mu$  as the first and second component. For the third, we use the composition of the comparison maps

$$h : H^\iota \otimes \mathbb{Q}_\ell \cong H^{et} \cong H^\mu \otimes \mathbb{Q}_\ell$$

The map  $p$  induces a homomorphism  $\text{End}^\vee(H^\iota) \rightarrow \text{End}^\vee(H^{\mathcal{T}})$ . We claim that this gives an isomorphism. Here we use the duality principle given in lemma 2.1.2. The dual of  $p$  is given by  $p^*(f) = 1_p \diamond f$ . The map is injective as it has a left inverse given by

$$g \mapsto (\gamma \diamond 1_H) \circ (1_q \diamond g) \circ (\gamma^{-1} \diamond 1_H)$$

The map  $p^*$  is also surjective, because

$$p^*(1_q \diamond [(\eta \diamond 1) \circ g \circ (\eta^{-1} \diamond 1)]) = g$$

By an identical argument  $\text{End}^\vee(H^\mu) \cong \text{End}^\vee(H^{\mathcal{T}})$

□

### 3.5 Base change

**Lemma 3.5.1.** *Let  $S = S_1 \cup S_2 \cup \dots \cup S_n$  be a decomposition into connected components. Choose stratifications  $\Sigma_i$  of  $S_i$  and let  $\Sigma = \bigcup \Sigma_i$ . Then*

$$\mathcal{PM}(S, \Sigma) = \mathcal{PM}(S_1, \Sigma_1) \times \mathcal{PM}(S_2, \Sigma_2) \times \dots \times \mathcal{PM}(S_n, \Sigma_n)$$

*Proof.* This is an immediate consequence of lemma 2.2.13, which applies because the empty families  $(\emptyset, \emptyset, i, w) \in \Delta(S_i, \Sigma_i)^{op}$  map to 0 under  $H$ .  $\square$

A morphism of (pointed) stratified varieties is a morphism of varieties such that a nonempty preimage of any stratum is a union of strata (and base points go to base points). If the underlying morphism of varieties is the identity, we say that the first stratification refines the second.

**Construction 3.5.2.** *Suppose that  $f : (T, \Lambda, t) \rightarrow (S, \Sigma, s)$  is a morphism of pointed stratified varieties. First, suppose  $(*)$  that the map on base points is surjective. Applying corollary 2.2.10 to*

$$(X \rightarrow S, Y, i, w) \mapsto h_T^i(X \times_S T, Y \times_S T)(w)$$

*yields an extension, which is the base change functor  $f^* : \mathcal{PM}(S, \Sigma, s) \rightarrow \mathcal{PM}(T, \Lambda, t)$ . If  $f(t) \neq s$ , set  $T' = T \coprod s'$  where  $s' = s - f(t)$ . Then the map*

$$f' : (T', \Lambda' = \Lambda \cup s', t' = t \cup s') \rightarrow (S, \Sigma, s),$$

*which is  $f$  on  $T$  and identity on  $s'$ , satisfies  $(*)$ . We now define  $f^*$  as the composite*

$$\mathcal{PM}(S, \Sigma, s) \xrightarrow{f'^*} \mathcal{PM}(T', \Lambda', t') = \mathcal{PM}(T, \Lambda, t) \times \mathcal{PM}(s', s', s') \xrightarrow{p} \mathcal{PM}(T, \Lambda, t)$$

*where  $p$  is the projection.*

We can always extend a morphism of stratified varieties to a morphism of pointed stratified varieties, and this way obtain base change functor  $f^* : \mathcal{PM}(S, \Sigma) \rightarrow \mathcal{PM}(T, \Lambda)$ . Alternately, we could define this directly in the spirit of remark 3.4.3 without recourse to base points. We mention two important special cases of this construction:

1. The construction applies when  $S = T$  and  $\Lambda$  refines  $\Sigma$ . This leads to faithful exact embeddings  $\rho_{\Sigma, \Lambda} : \mathcal{PM}(S, \Sigma) \rightarrow \mathcal{PM}(S, \Lambda)$ .
2. When  $T = s$  is the set of, say  $n$ , base points, we get an embedding  $\mathcal{PM}(S, \Sigma) \rightarrow \mathcal{PM}(k)^n$ .

The last map is usually not an equivalence.

**Example 3.5.3.** *Let  $S = \mathbb{A}^1$  with  $\Sigma = \{0, \mathbb{A}^1 - \{0\}\}$  and base points  $s = \{0, 1\}$ . Now consider the motive  $\mathbb{Q}_S$  represented by  $(id : S \rightarrow S, \emptyset, 0, 0)$ . By passing to sheaves under  $R_B$ , we see that this is not isomorphic to  $\mathbb{Q}_0 \oplus \mathbb{Q}_1$ , so the base point map  $\mathcal{PM}(S) \rightarrow \mathcal{PM}(k) \times \mathcal{PM}(k)$  cannot be an equivalence.*

When  $f : T \rightarrow S$  is an inclusion, we often denote  $f^*M$  by  $M|_T$ .

**Lemma 3.5.4.** *The assignment  $(S, \Sigma) \mapsto \mathcal{PM}(S, \Sigma)$ ,  $f \mapsto f^*$  yields a contravariant pseudofunctor from the category of stratified varieties to the 2-category of abelian categories. This commutes with  $R_B$ .*

*Proof.* The functor  $id_S^* : \mathcal{PM}(S, \Sigma) \rightarrow \mathcal{PM}(S, \Sigma)$  is the extension of  $(X \rightarrow S, Y, i, w) \mapsto h_S^i(X, Y)(w)$ . So clearly we have a natural isomorphism  $id_S^* \cong id_{\mathcal{PM}(S)}$ . Suppose that  $f : (T, \Lambda) \rightarrow (S, \Sigma)$  and  $g : (V, \Theta) \rightarrow (T, \Lambda)$  are given, and assume the maps surject on base points. Then  $(f \circ g)^*$  is the extension of

$$(X \rightarrow S, Y, i, w) \mapsto h_V^i(X \times_S V, Y \times_S V)(w) \cong g^* h_T^i(X \times_S T, Y \times_S T)(w)$$

So we obtain a natural isomorphism  $(f \circ g)^* \cong g^* \circ f^*$ . These natural transformations can be seen to have the necessary compatibilities (see appendix A) to define a pseudofunctor.

The last statement follows from the isomorphism

$$R_B(h_T^i(X \times_S T, Y \times_S T)(w)) \cong H_T^i(X \times_S T, Y \times_S T)$$

and corollary 2.2.11. □

**Definition 3.5.5.** *The category of motivic sheaves of  $F$ -modules is given by the 2-colimit*

$$\mathcal{PM}(S; F) = 2\text{-}\varinjlim_{\Sigma} \mathcal{PM}(S, \Sigma; F)$$

(see appendix A).

It follows from the discussion in appendix A that  $\mathcal{PM}(S; F)$  is abelian, and the natural maps  $\mathcal{PM}(S, \Sigma; F) \rightarrow \mathcal{PM}(S; F)$  are exact.

Note that  $h_S^i(X, Y) \in \mathcal{PM}(S, \Sigma)$  maps to the same symbol  $h_S^i(X, Y)$  under refinement. We denote the common value in the colimit by  $h_S^i(X, Y)$  as well. Observe that  $R_B$  provides a faithful exact embedding of  $\mathcal{PM}(S, \Sigma)$  into the category  $Sh(S)$  of sheaves of  $F$ -vector spaces on  $S$ . This is compatible with refinement. So it passes to the limit. Therefore in more concrete terms, we can identify  $\mathcal{PM}(S)$  (up to equivalence) with the directed union of subcategories

$$\bigcup_{\Sigma} R_B(\mathcal{PM}(S, \Sigma)) \subset Sh(S)$$

Note that this lies in the subcategory  $Cons(S)$  of constructible sheaves.

As a corollary to lemma 3.5.4.

**Corollary 3.5.6.**  *$S \mapsto \mathcal{PM}(S)$  is a contravariant pseudofunctor.*

There are a number of useful variations of this construction. Let  $\Delta_c(S, \Sigma) \subset \Delta(S, \Sigma)$  denote the full subgraph consisting of tuples  $(X \rightarrow S, Y, i, w)$  where  $X \rightarrow S$  is projective. Then set  $\mathcal{PM}_c(S, \Sigma)$  be the category of comodules over  $End^v(H_{\Sigma}|_{\Delta_c(S, \Sigma)})$ . The category of compact motives

$$\mathcal{PM}_c(S) = 2\text{-}\varinjlim_{\Sigma} \mathcal{PM}_c(S, \Sigma; F)$$

This can be regarded as an abelian subcategory of  $\mathcal{PM}(S)$ . It contains motives  $h_S^i(X)$  of projective families.

## 4 Motivic Sheaves

### 4.1 Zariski Descent

In the next section, we will see that motivic sheaves in  $\mathcal{PM}_c(S)$  can be patched on a Zariski open cover. This is not yet evident for  $\mathcal{PM}(S)$ , and may be an indication of a defect in the definition. Since this issue plays a relatively minor role (so far), we just sketch a construction of a new category of motivic sheaves  $\mathcal{M}(S)$  which removes this defect.

We can repackage  $\mathcal{M}(S)$  in the language of fibred categories [Gi, Vi]. Given a functor  $\pi : \mathcal{F} \rightarrow \mathcal{C}$ , the fibre  $\pi^{-1}(A)$  over  $A \in \text{Ob}(\mathcal{C})$  is the category with objects  $\pi^{-1}A$  and morphism  $\pi^{-1}id_A$ . Fibres need not behave as expected, for example fibres over isomorphic objects need not be equivalent, unless further conditions are imposed. An arrow  $\phi \in \text{Mor}\mathcal{F}$  is *cartesian* if for any commutative diagram consisting of solid arrows

$$\begin{array}{ccccc}
 A & & & & \\
 \swarrow \psi & & & & \\
 & B & \xrightarrow{\phi} & C & \\
 \swarrow \pi(\psi) & & & & \\
 \pi(A) & & & & \\
 \swarrow & & & & \\
 & \pi(B) & \xrightarrow{\pi(\phi)} & \pi(C) & 
 \end{array}$$

the dotted arrow can be filled in uniquely. The functor  $\pi : \mathcal{F} \rightarrow \mathcal{C}$  is *fibred* if any arrow of  $\mathcal{C}$  can be lifted to a cartesian arrow of  $\mathcal{F}$  with a specified target. It is sometimes convenient to fix a collection of specific cartesian lifts. Such a collection is called a *cleavage*. Given a cleavage, there is a well defined way to define a pullback functor  $f^* : \pi^{-1}(B) \rightarrow \pi^{-1}(A)$  for any  $f : A \rightarrow B \in \text{Mor}\mathcal{C}$ . These form a pseudofunctor. Conversely, any pseudofunctor determines a fibred category. Define  $\mathcal{M}$  to be the category whose objects are pairs  $(S \in \text{Var}_k, M \in \mathcal{M}(S))$ , and morphisms are pairs  $(f : T \rightarrow S, M \rightarrow f^*N \in \text{Mor}\mathcal{M}(T))$ . This is fibred over  $\text{Var}_k$  via the natural projection  $\pi : \mathcal{M} \rightarrow \text{Var}_k$ . The categories  $\mathcal{M}(S)$  are just the fibres  $\pi^{-1}(S)$ , and the original pseudofunctor is determined by the cleavage  $\{(f, f^*N = f^*N)\}$ .

**Definition 4.1.1.** *Let  $\pi : \mathcal{M} \rightarrow \text{Var}_k$  denote that stack associated to  $\mathcal{PM} \rightarrow \text{Var}_k$  for the Zariski site [Gi, chap 2 §2]. The category of motivic sheaves  $\mathcal{M}(S)$  over  $S \in \text{ObVar}_k$  is the fibre over  $S$ .*

Unravelling all of this, we see that:

1.  $\mathcal{M} \rightarrow \text{Var}_k$  is a fibred category. Fix a cleavage for it, so that there are functors  $f^* : \mathcal{M}(S) \rightarrow \mathcal{M}(T)$  for each  $f : T \rightarrow S$ . If  $f$  is an inclusion, we denote  $f^*M$  by  $M|_S$ .
2. For any  $M, N \in \mathcal{M}(S)$ ,  $U \mapsto \text{Hom}_{\mathcal{M}(S)}(M|_U, N|_U)$  is a sheaf on the Zariski topology, i.e.  $\mathcal{M}$  is a prestack.

3. Given an open cover  $\{U_i\}$  of  $S$ ,  $M_i \in \mathcal{PM}(U_i)$  with isomorphisms  $g_{ji} : M_i|_{U_{ij}} \cong M_j|_{U_{ij}}$  satisfying the usual cocycle condition  $g_{ii} = id$ ,  $g_{ik} = g_{ij}g_{jk}$ , there exists  $M \in \mathcal{M}(S)$  such that  $M_i \cong M|_{U_i}$ .
4. There is a functor  $\iota : \mathcal{PM} \rightarrow \mathcal{M}$  of fibred categories. This means that it fits into a commutative diagram

$$\begin{array}{ccc}
 \mathcal{PM} & \xrightarrow{\iota} & \mathcal{M} \\
 & \searrow & \swarrow \\
 & \text{Var}_k & 
 \end{array}$$

and takes cartesian arrows to cartesian arrows. The functor  $\iota$  is universal among all such functors from  $\mathcal{PM}$  to stacks. So that any functor  $\mathcal{M} \rightarrow \mathcal{M}'$  to a fibred category satisfying conditions (2) and (3) must factor through  $\mathcal{M}$  in an essentially unique way, or more precisely, there is an equivalence between the categories  $Hom_{fibcat}(\mathcal{M}, \mathcal{M}')$  and  $Hom_{fibcat}(\mathcal{M}, \mathcal{M}')$ .

A concrete construction of the associated stack is given in [LM, chap 3] (although stated for groupoids, it works in general). It is a two step process. First, one constructs a prestack  $\mathcal{PM}^+$ . The objects of  $\mathcal{PM}^+(S)$  are the same as for  $\mathcal{PM}(S)$ . For  $Hom_{\mathcal{PM}^+(S)}(M, N)$  we take the global sections of the sheaf associated to  $U \mapsto Hom_{\mathcal{PM}(S)}(M|_U, N|_U)$ . This forces condition (2) to hold. Now construct  $\mathcal{M}(S)$  as the category whose objects consists of descent data, i.e., collections  $(M_i \in \mathcal{PM}^+(U_i), g_{ij})$  as in (3). Given two objects  $(M_i \in \mathcal{PM}^+(U_i), g_{ij}), (M'_i \in \mathcal{PM}^+(U'_i), g'_{ij})$ , after passing to a common refinement we can assume that  $U_i = U'_i$ . A morphism is then given by a compatible collection of morphisms  $M_i \rightarrow M'_i$ . We have obvious functors  $\mathcal{PM}(S) \rightarrow \mathcal{PM}^+(S) \rightarrow \mathcal{M}(S)$ . It is easy to see from this description that  $\mathcal{M}(S)$  is abelian, and  $\mathcal{PM}(S) \rightarrow \mathcal{M}(S)$  is exact. To summarize:

**Theorem 4.1.2.**  *$\mathcal{M}$  is a stack of abelian categories over  $Var_k$ . There is an exact functor  $\mathcal{PM} \rightarrow \mathcal{M}$  which is universal among all functors from  $\mathcal{PM}$  to stacks.*

As corollary, the realization functors  $R_B$  et cetera extend to  $\mathcal{M}$ .

We define  $h_S^i(X, Y)(w)$  as the image of  $h_S^i(X, Y)(w)$  in  $\mathcal{M}(S)$ . Since the collection of sheaves  $Sh(S)$  forms a stack over  $S_{zar}$ , we can see that  $R_B$  factors through  $\mathcal{M}$ . With the help of the above explicit description, we get a slightly sharper result.

**Corollary 4.1.3.** *The functor  $R_B : \mathcal{M}(S) \rightarrow Sh(S)$  extends to an exact faithful functor  $R_B : \mathcal{M}(S) \rightarrow Sh(S)$ . In particular,  $R_B(h_S^i(X, Y)(w)) = H_S^i(X, Y)$ .*

By a very similar argument, we have:

**Corollary 4.1.4.** *The functor  $R_{et} : \mathcal{M}(S, F) \rightarrow Sh(S_{et}, F)$  extends to an exact faithful functor  $R_{et} : \mathcal{M}(S, F) \rightarrow Sh(S_{et}, F)$ .*

We want give an alternative concrete construction. It will be convenient to start with some generalities. It will be convenient to “sheafify” the discussion of section 2.2. Let  $\mathcal{D} = \{D_U\}$  be a collection of graphs indexed Zariski open subsets of  $S$ , such that for each inclusion  $\iota : V \subset U$  we have a restriction morphism  $\iota^{-1} : D_V \rightarrow D_U$  satisfying  $(\iota \circ \mu)^{-1} = \mu^{-1} \circ \iota^{-1}$ . Let  $H : D_U \rightarrow F\text{-mod}$  be a collection of morphisms compatible with restriction. We refer  $(\mathcal{D}, H)$  as a compatible collection. Given such a collection, let  $\mathcal{E}^\vee(H)$  to be the sheaf associated to the presheaf

$$U \mapsto \text{End}^\vee(H|_{D_U})$$

Let  $\mathcal{E}^\vee(H)\text{-comod}$  denote the category of Zariski sheaves of finite dimensional  $F$ -vector spaces with a coaction by  $\mathcal{E}^\vee(H)$ . When  $\mathcal{D}$  consists of finite graphs, this is isomorphic to the category of modules over the sheaf of rings

$$\mathcal{E}(H) = \mathcal{H}om(\mathcal{E}^\vee(H), F_S)$$

Let  $\text{Coh}(\mathcal{E}(H))$  denote the category of coherent i.e. locally finitely presented, modules. The colimit

$$\text{Coh}(\mathcal{E}^\vee(H)) = \varinjlim_{\mathcal{D}' \subset \mathcal{D} \text{ finite}} \text{Coh}(\mathcal{E}(H|_{\mathcal{D}'}))$$

can be realized as a subcategory of  $\mathcal{E}^\vee(H)\text{-comod}$ .

Suppose that we are given compatible collections  $\tilde{\mathcal{D}}$  and  $\mathcal{D}$  and morphisms  $\pi : \tilde{D}_U \rightarrow D_u$  compatible with restriction. Then a functor  $H$  on  $\mathcal{D}$  induces a functor

$$\text{Coh}(\mathcal{E}^\vee(H)) \rightarrow \text{Coh}(\mathcal{E}^\vee(H \circ \pi))$$

An analogue of corollary 2.2.4 is:

**Lemma 4.1.5.** *This functor is an equivalence if for any two objects of the fiber  $\pi : \tilde{D}_U \rightarrow D_u$  are connected by a chain of morphisms on some cover of  $U$ .*

Given a functor  $H^\# : \Delta(S, \Sigma) \rightarrow F\text{-mod}^2$  compatible with restriction and such that  $H = p_1 \circ H^\#$ , we can define a category  $\mathcal{M}^\#(S, \Sigma)$  by mimicing the procedure used to define  $\mathcal{M}'$ . We have refinement of corollary 2.3.2 and which follows by the same argument.

**Lemma 4.1.6.**  $\mathcal{M}^\#(S, \Sigma) / \ker p_1 \sim \mathcal{M}(S, \Sigma)$ .

Returning to the initial set up of a stratified variety We now give the explicit construction. Fix a stratified variety  $(S, \Sigma)$  with a given set of base points  $s$  for  $\Sigma$ . Given Zariski open sets  $V \subset U \subset S$ , let  $s'$  denote the intersection of  $s$  with  $U - V$ . As in construction 3.5.2, we have a map given by composition

$$\text{End}^\vee(H|_{\Delta(U, \Sigma)}) \rightarrow \text{End}^\vee(H|_{\Delta(V, \Sigma)}) \times \text{End}^\vee(H|_{\Delta(s')}) \rightarrow \text{End}^\vee(H|_{\Delta(V, \Sigma)})$$

where the last map is projection. This makes

$$U \mapsto \text{End}^\vee(H|_{\Delta(U, \Sigma)})$$



into a presheaf of coalgebras. Let  $\mathcal{E}^\vee(S, \Sigma)$  denote the associated sheaf on the Zariski topology  $S_{zar}$ .

Then  $\mathcal{D} = \{\Delta(U, \Sigma)\}$  gives a compatible collection with restrictions given by

$$(X \rightarrow V, Y, i, w) \mapsto (X \times_V U, Y \times_V U, i, w)$$

Set  $Coh(\mathcal{E}^\vee(S, \Sigma)) = Coh(\mathcal{E}^\vee(H))$  as above. A premotivic sheaf  $M$  in  $\mathcal{PM}(S, \Sigma)$  determines an object of  $U \mapsto M|_U$  of  $\mathcal{E}^\vee(S, \Sigma)$ -comod, which can be seen to lie in some  $Coh(\mathcal{E}(H|_{\mathcal{D}'}))$ , with  $\mathcal{D}' \subset \mathcal{D}$  finite, and therefore in  $Coh(\mathcal{E}^\vee(S, \Sigma))$ . This gives a fully faithful exact functor  $\mathcal{PM}(S, \Sigma) \rightarrow Coh(\mathcal{E}^\vee(S, \Sigma))$ .

**Lemma 4.1.7.**  *$Coh(\mathcal{E}^\vee(S, \Sigma))$ -comod coincides with the full subcategory of comodules  $M$  which are locally isomorphic to objects of  $\mathcal{PM}(-, \Sigma)$*

*Proof.* An object of  $Coh(\mathcal{E}^\vee(S, \Sigma))$ -comod lies in some  $Coh(\mathcal{E}(\mathcal{D}))$ . So it is locally of the form  $coker(\mathcal{E}(\mathcal{D})^n \rightarrow \mathcal{E}(\mathcal{D})^m)$  which lies in  $\mathcal{PM}(S, \Sigma)$ .  $\square$

Set  $\mathcal{M}'(S) = 2\text{-}\varinjlim_{\Sigma} Coh(\mathcal{E}^\vee(S, \Sigma))$ . Then we have a functor  $\mathcal{PM}(S) \rightarrow \mathcal{M}'(S)$  induced from the one above.

**Lemma 4.1.8.**  *$\mathcal{M}(S)$  is equivalent to  $\mathcal{M}'(S)$ .*

*Proof.* The functor  $\mathcal{PM}(S) \rightarrow \mathcal{M}'(S)$  induces a functor  $\mathcal{PM}^+(S) \rightarrow \mathcal{M}'(S)$  which is fully faithful. This extends to a functor  $\mathcal{M}(S) \rightarrow \mathcal{M}'(S)$  which is again fully faithful. It also essentially surjective by the previous lemma.  $\square$

In view of this result, we will generally denote  $\mathcal{M}'(\dots)$  simply by  $\mathcal{M}(\dots)$  from henceforth. We also define  $\mathcal{M}(S, \Sigma) = Coh(\mathcal{E}^\vee(S, \Sigma))$ -comod.

## 4.2 Extension by zero

Let  $j : S \hookrightarrow \bar{S}$  be an open immersion with boundary  $\partial\bar{S} = \bar{S} - S$ . Suppose that  $\Sigma$  is a stratification of  $\bar{S}$  such that  $S$  is union of strata. Let  $\Delta(\bar{S}, \partial\bar{S}, \Sigma) \subset \Delta(\bar{S}, \Sigma)$  denote the full subgraph consisting of tuples  $(\bar{X} \rightarrow \bar{S}, \bar{Y}, i, w)$  where  $\bar{Y} \supseteq f^{-1}\partial\bar{S}$ . We construct the category  $\mathcal{M}(\bar{S}, \partial\bar{S}, \Sigma)$  as in §3.3, as the category of  $End^\vee(H_\Sigma|_{\Delta(\bar{S}, \partial\bar{S}, \Sigma)})$ -comodules, and  $\mathcal{M}(\bar{S}, \partial\bar{S})$  is the colimit of these over  $\Sigma$ . There is an exact faithful forgetful functor  $\iota_{\bar{S}} : \mathcal{M}(\bar{S}, \partial\bar{S}) \rightarrow \mathcal{M}(\bar{S})$ . We define  $\Delta_{ex}(S, \Sigma) \subseteq \Delta(S, \Sigma)$  be the full subgraph of tuples  $(X \rightarrow S, Y, i, w)$  which extend to  $\Delta(\bar{S}, \partial\bar{S}, \Sigma)$ . Then we can define the subcategory  $\mathcal{M}_{ex}(S, \Sigma) \subseteq \mathcal{M}(S, \Sigma)$  by taking comodules over  $H_\Sigma$  restricted to  $\Delta_{ex}$ . Let

$$\mathcal{M}_{ex}(S) = 2\text{-}\varinjlim \mathcal{M}_{ex}(S, \Sigma) \subset \mathcal{M}(S)$$

**Lemma 4.2.1.** *The category  $\mathcal{M}(\bar{S}, \partial\bar{S})$  is equivalent to a subcategory  $\mathcal{M}_{ex}(S) \subset \mathcal{M}(S)$  via  $j^*$ .*

*Proof.* We start by proving that  $j^* : \mathcal{M}(\bar{S}, \partial\bar{S}, \Sigma) \rightarrow \mathcal{M}_{ex}(S, \Sigma)$  is an equivalence. We have a morphism  $\pi : \Delta(\bar{S}, \partial\bar{S}, \Sigma) \rightarrow \Delta_{ex}$  given by restriction. This is surjective by definition, and the fibres of  $\pi$  are clearly connected. For  $(\bar{X} \rightarrow$

$\bar{S}, \bar{Y}, i, w) \in \Delta(\bar{S}, \partial\bar{S}, \Sigma)$ ,  $R^i j_{\bar{X}, \bar{Y}}! F = 0$  outside of  $S$ . Thus  $H(\bar{X} \rightarrow \bar{S}, \bar{Y}, i, w)$  coincides with the value of  $H$  on the restriction. Therefore we have a morphism of pairs  $(\Delta(\bar{S}, \partial\bar{S}, \Sigma), H) \rightarrow (\Delta_{ex}, H)$ , and so  $j^* : \mathcal{M}(\bar{S}, \partial\bar{S}, \Sigma) \rightarrow \mathcal{M}_{ex}(S, \Sigma)$  is an equivalence by corollary 2.2.4. Passing to the limit yields the lemma.  $\square$

**Lemma 4.2.2.**  $\mathcal{M}_c(S) \subseteq \mathcal{M}_{ex}(S)$ .

*Proof.* Given a projective family  $X \subset \mathbb{P}^N \times S$  over  $S$ , we get an extension over  $\bar{S}$  by taking the closure  $\bar{X} \subset \mathbb{P}^N \times \bar{S}$ . Given  $Y \subset X$ , we may take  $\bar{Y} \subset \bar{X}$  to be the union of the closure of  $Y$  with the preimage of  $\partial S$ . Thus we see that  $\Delta_c(S, \Sigma) \subset \Delta_{ex}(S, \Sigma)$ .  $\square$

**Definition 4.2.3.** Define  $j_! : \mathcal{M}_{ex}(S) \rightarrow \mathcal{M}(\bar{S})$  by  $j_! = \iota_{\bar{S}} \circ j'^{* - 1}$ .

**Lemma 4.2.4.** This is compatible with extension by zero for sheaves  $j_! : Sh(S) \rightarrow Sh(\bar{S})$  in the sense that  $R_B(j_! \mathcal{F}) = j_! R_B(\mathcal{F})$ . We have  $j_! \mathcal{M}_c(S) \subset \mathcal{M}_c(\bar{S})$ .

**Proposition 4.2.5.** There exists a natural transformation  $j_! j^* \rightarrow 1$  on  $\mathcal{M}(\bar{S})$  compatible with the usual adjunction map for sheaves.

*Proof.* The proof is similar to the proof of proposition 5.1.4. Let  $Mor' \subset Mor\mathcal{M}(\bar{S})$  denote the subcategory of morphisms  $M_2 \rightarrow M_1$  such that there is a commutative diagram

$$\begin{array}{ccc} j_! j^* R_B M_1 & \longrightarrow & R_B M_1 \\ \downarrow \cong & & \downarrow = \\ R_B M_2 & \longrightarrow & R_B M_2 \end{array}$$

commutes. The functor  $(M_2 \rightarrow M_1) \mapsto M_1$  is clearly faithful and exact. Therefore by corollary 2.2.10, we get a functor  $\mathcal{M}(\bar{S}) \rightarrow Mor'$  such that

$$h_{\bar{S}}^i(X, Y)(w) \mapsto [h_{\bar{S}}^i(X, Y \cup f^{-1} \partial \bar{S})(w) \rightarrow h_{\bar{S}}^i(X, Y)(w)]$$

for each  $(f : X \rightarrow \bar{S}, Y, i, w) \in \Delta(S)$ .  $\square$

**Proposition 4.2.6.**  $\mathcal{PM}_c(S)$  forms a stack in the Zariski topology, i.e. objects and morphisms can be patched on Zariski open covers.

*Proof.* By compactness and induction, it suffices to treat covers  $S = U_0 \cup U_1$  consisting of two open sets. Given  $M_i \in Ob\mathcal{PM}_c(U_i)$  with an isomorphism  $f : M_0|_{U_0 \cap U_1} \cong M_1|_{U_0 \cap U_1}$ . Let  $j_i : U_i \hookrightarrow S$  and  $j_{01} : U_0 \cap U_1 \rightarrow S$  denote the inclusions. We define a morphism

$$g : j_{01}! M_0 \rightarrow j_{01}! M_0 \oplus j_{1!} M_1$$

extending 1 on the first factor and  $-f$  on the second. Then  $M = \text{coker}(g)$  gives an object of  $\mathcal{PM}_c(S)$  which restricts to  $M_i$ .  $\square$

### 4.3 Cellular decompositions

A number of constructions will be based on the existence “cellular” decompositions. The use of such decompositions plays a key role in Nori’s work and also [Ar]. This hinges on a result of Beilinson [B2] that Nori calls the basic lemma. We need a slight modification of this result. Define a map  $f : X \rightarrow S$  to be equidimensional (respectively uniform) if  $\dim X_s$  is constant (respectively, if for every  $s$ , all irreducible components of  $X_s$  have the same dimension).

**Proposition 4.3.1.** *Let  $X \rightarrow S$  be a uniform affine morphism. Suppose that  $\mathcal{F}$  is a constructible sheaf on  $X$  such that  $(X \rightarrow S, \mathcal{F})$  is controlled. Then there exists a Zariski open cover  $\{S^\beta\}$  of  $S$ , dense affine open subsets  $g^\beta : U^\beta \hookrightarrow X^\beta = f^{-1}S^\beta$  such that*

- (1)  $(U^\beta \rightarrow S^\beta, g_!^\beta g^{\beta*} \mathcal{F})$  is controlled.
- (2) For every  $s \in S^\beta$ ,

$$H_{S^\beta}^i(X^\beta, X^\beta - U^\beta; \mathcal{F})_s = H_{S^\beta}^i(X^\beta, g_!^\beta g^{\beta*} \mathcal{F})_s = 0$$

unless  $i = \dim X_s$ .

*Proof.* Most of the argument is pretty much identical to the proof of [B2, lemma 3.3]. Nevertheless, we spell this out since the result is central. Let

$$\begin{array}{ccccc} X & \xrightarrow{g} & T & \xrightarrow{h} & S \\ \downarrow k & \nearrow \bar{g} & & \nearrow \bar{f} & \\ \bar{X} & & & & \end{array}$$

be a control diagram. After replacing  $\bar{X}$  by the blow up along  $\bar{X} - X$ , we can assume that  $k$  is affine. We can choose a divisor  $Z' \subset T$  such that  $\mathcal{F}$  is constant on the complement of  $Z = g^{-1}Z'$ . Let  $\ell : X - Z \hookrightarrow X$  be the inclusion, then  $M = \ell_! \mathcal{F}_{X-Z}$  is also controlled by the above diagram. Set  $\bar{M} = \mathbb{R}k_* M$ . Note that  $\mathcal{F}[\dim X_s]$ ,  $M[\dim X_s]$  and  $\bar{M}[\dim X_s]$  restrict to perverse sheaves on the fibres of  $f$  or  $\bar{f}$ , because  $\ell$  and  $k$  are affine embeddings [BBD, cor. 4.1.3].

Fix an embedding  $\bar{X} \subset \mathbb{P}^N \times S$  over  $S$ . Given a hyperplane  $H$ , let

$$\begin{array}{ccccc} V = \bar{X} - H & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & H \\ \uparrow k' & & \uparrow k & & \uparrow k'' \\ X - H = V \cap X & \xrightarrow{j'} & X & \xleftarrow{i'} & H \cap X \end{array}$$

denote the inclusions. Let  $j_s : \bar{X}_s - H \hookrightarrow \bar{X}_s$  etc. denote the restrictions of these inclusions to the fibre over  $s \in S$ . We claim that

$$j_! \mathbb{R}k'_* M|_{V \cap X} \cong \mathbb{R}k_* j'_! M|_{V \cap X} \quad (7)$$

holds for a dense open set  $P$  of hyperplanes  $H$  in the dual projective space  $\check{\mathbb{P}} = \check{\mathbb{P}}^N$ . The argument which we sketch is from [B2, pp 35-36]. First note that (7) equivalent to

$$i^* \mathbb{R}k_* M \cong \mathbb{R}k''_* i'^* M \quad (8)$$

by the argument indicated in [loc. cit.]. Let  $H_P \subset \bar{X}_P = \bar{X} \times \check{\mathbb{P}}$  denote the universal hyperplane. Let  $M_P$  denote the pullback of  $M$  to  $X_P = X \times \check{\mathbb{P}}$ . Let  $i_P : H_P \rightarrow \bar{X}_P$  denote the canonical morphism. Similarly the other morphism  $i', \dots$  have obvious extensions denoted by  $(-)_P$  such that they can be recovered by taking the fibre at  $H \in \check{\mathbb{P}}$ . With this notation, it suffices to prove

$$i_P^* \mathbb{R}k_{P*} M_P \cong \mathbb{R}k''_{P*} i'_P{}^* M_P$$

by virtue of theorem 3.1.10 (2). But this is a consequence of the 3.1.10 (3), because  $i_P$  is locally trivial.

By combining this with proposition 3.1.8, we can find a cover  $\{S^\beta\}$  and hyperplanes  $H^\beta \in P$  so that  $j'_! M|_{V \cap X}$  is controlled over  $S^\beta$ . In order to simplify notation, replace  $S$  by  $S^\beta$  etc. below, for a fixed  $\beta$ . We set  $U = X - H - Z$  with inclusion  $g$ . Then  $g_! g^* \mathcal{F} = j'_! M|_{V \cap X}$ . So the first item of the proposition holds, and in particular, this sheaf has the base change property.

It remains to prove item (2). Since each fibre  $X_s$  is affine, we have  $H_S^i(g_! g^* \mathcal{F}) = 0$  for  $i > \dim X_s$  by Artin's vanishing theorem. The remaining half also follows from the affineness of  $X$ . As  $f_s$  is affine,  $\mathbb{R}f_{s!}$  is left  $t$ -exact for the perverse  $t$ -structure [BBD, cor. 4.1.2]. Therefore

$$H_c^i(V \cap X_s, \bar{M}|_V[\dim X_s]) = \mathcal{H}^i(\mathbb{R}f_{s!} \bar{M}|_V[\dim X_s]) = 0$$

for  $i < 0$ . Then from this, (7) and the proper base change theorem we obtain,

$$\begin{aligned} R^i f_* (j'_! M|_{V \cap X})_s &\cong \mathcal{H}^i(\mathbb{R}f_* j'_! M|_{V \cap X})_s \\ &\cong \mathcal{H}^i(\mathbb{R}\bar{f}_* \mathbb{R}k_* j'_! M|_{V \cap X})_s \\ &\cong \mathcal{H}^i(\mathbb{R}\bar{f}_* j'_! \mathbb{R}k_* M|_{V \cap X})_s \\ &\cong \mathcal{H}^i(\mathbb{R}\bar{f}_* j'_! M|_{V \cap X})_s \\ &\cong \mathcal{H}^i(\mathbb{R}\bar{f}_* j'_! \bar{M}|_V)_s \\ &\cong (R^i(\bar{f} \circ j)_! \bar{M}|_V)_s \\ &= 0 \end{aligned}$$

for  $s \in S$  and  $i < \dim X_s$ . □

**Corollary 4.3.2.** *The schemes  $U^\beta \rightarrow S^\beta$  can be assumed to be smooth.*

**Corollary 4.3.3.** *If  $X \rightarrow S$  is equidimensional of relative dimension  $n$ . Then*

$$H_{S^\beta}^i(X, g_! g^{\beta*} \mathcal{F}) = 0$$

*unless  $i = n$ .*

We define  $\Delta_{eq}(S, \Sigma) \subset \Delta(S, \Sigma)$  by requiring  $X \rightarrow S$  and  $Y \rightarrow S$  to be equidimensional. The categories  $\mathcal{M}_{eq}(S, \Sigma)$  and  $\mathcal{M}_{eq}(S)$  are defined by the same procedure as before by restricting  $H$  to  $\Delta_{eq}$ . These can be viewed as subcategories of  $\mathcal{M}(S)$ .

**Lemma 4.3.4.** *Let  $X \rightarrow S$  be an affine equidimensional morphism to a variety which is controlled with respect to a sheaf  $\mathcal{F}$ . Then there is a Zariski open cover  $\{S^\beta\}$  of  $S$  and filtrations*

$$X_0^\beta \subset X_1^\beta \subset \dots \subset X_n^\beta = X^\beta = f^{-1}S^\beta$$

such that

- (1)  $X_i^\beta \rightarrow S^\beta$  is equidimensional of pure relative dimension  $i$ .
- (2) The pairs  $(X_a^\beta \rightarrow S, X_{a-1}^\beta)$  are controlled with respect to  $\mathcal{F}$ , and

$$H_{S^\beta}^i(X_a^\beta, X_{a-1}^\beta; \mathcal{F}) = 0$$

for  $i \neq a$ .

If in addition,

$$X'_0 \subset X'_1 \subset \dots \subset X'_n = X$$

is a given chain of closed sets each of pure relative dimension  $i$ . Then we can choose  $X'_i \cap X^\beta \subseteq X_i^\beta$ .

*Proof.* This follows from the previous proposition and induction on  $\dim X$ .  $\square$

**Definition 4.3.5.** *Suppose that we are given a morphism  $\tilde{X} \rightarrow X$  of  $S$ -schemes, a Zariski open cover  $\{S^\beta\}$  of  $S$ , filtrations by closed sets of the preimage of each  $S^\beta$*

$$\tilde{X}^\beta = \tilde{X}_d^\beta \supset \tilde{X}_{d-1}^\beta \supset \dots \supset \tilde{X}_0^\beta \supset \tilde{X}_{-1}^\beta = \emptyset$$

We refer to the collection  $(\dots, \tilde{X}_\bullet^\beta)$  as a quasi-filtration on  $X$ , and the whole thing as a quasi-filtered  $X$ -variety.

These objects form a category  $QVar_S$ , where a morphism

$$\phi : (\{S^\beta\}_{\beta \in B}, \tilde{X}' \rightarrow X, \tilde{X}'_\bullet^\beta) \rightarrow (\{T^\gamma\}_{\gamma \in G}, \tilde{X} \rightarrow X, \tilde{X}^\gamma)$$

is given by a map  $r : B \rightarrow G$ , such that  $S^\beta \subseteq T^{r(\beta)}$  plus commutative squares of  $S$ -schemes

$$\begin{array}{ccc} \tilde{X}' & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

with  $\tilde{X}'_\bullet^\beta$  mapping to  $\tilde{X}_\bullet^{r(\beta)}$ . We say that  $\phi$  covers  $f$ . Let say that the quasi-filtration is simple if the cover  $\{S^\beta\}$  consists of  $\{S\}$  alone. A filtered variety is

the special case of a simple quasi-filtered variety, where  $\tilde{X} \rightarrow X$  is the identity. Let  $FVar_S$  be the full subcategory of filtered varieties.

We give a relative version of Jouanolou's trick [Jo, lemma 1.5] below. To simply the statement, let us say that  $\tilde{X} \rightarrow X$  is *bundle of affine spaces* if  $\tilde{X} = T \times_{\text{Aff}(n)} \mathbb{A}_k^n$ , where  $T \rightarrow X$  is a torsor for the affine group in the Zariski topology

**Lemma 4.3.6.** *If  $f : X \rightarrow S$  is a quasi-projective morphism then there exists a commutative diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ & \searrow \tilde{f} & \downarrow f \\ & & S \end{array}$$

such that  $\tilde{f}$  is affine, and  $\pi$  is a bundle of affine spaces.

*Proof.* When  $X = \mathbb{P}^N \times S$ ,  $\tilde{X}$  can be taken to be product of  $S$  with the complement of the incidence variety  $St_N = \{(x, H) \in \mathbb{P}^N \times \mathbb{P}^N \mid x \notin H\}$ . For the general case, let  $X \subset \bar{X} \subset \mathbb{P}^N \times S$  be a relative compactification. After blowing up, we can assume that  $\bar{X} - X$  is a divisor. Then the preimage  $\tilde{X}$  of  $X$  in  $St_N \times S$  will do the job.  $\square$

When  $f : X \rightarrow S$  is projective, we see that the pullback of any constructible sheaf  $\pi^* \mathcal{F}$  is necessarily controlled. We say that a quasi-filtration  $(\pi : \tilde{X} \rightarrow X, \tilde{X}_\bullet)$  is *cellular* with respect to a controlled constructible sheaf  $\mathcal{F}$  if

1.  $\pi$  is a bundle of affine spaces,
2.  $\pi^* \mathcal{F}$  is controlled,
3.  $\tilde{X}_\bullet$  satisfies condition (2) of lemma 4.3.4 with respect to  $\pi^* \mathcal{F}$ , i.e.

$$H_{S^\beta}^i(\tilde{X}_a^\beta, \tilde{X}_{a-1}^\beta; \pi^* \mathcal{F}) = 0$$

Note that the first assumption implies that  $\mathbb{R}\pi_* \pi^* \mathcal{F} = \mathcal{F}$ . The second assumption can be seen to be redundant, but there is no harm in including it.

**Lemma 4.3.7.** *A bundle  $\pi : \tilde{X} \rightarrow X$  of affine spaces over an affine scheme admits a section.*

*Proof.* The bundle  $\tilde{X}/X$  is a homogeneous space associated to a torsor for the affine group  $\text{Aff}(n)$ . Using the exact sequence

$$1 \rightarrow G_a^n \rightarrow \text{Aff}(n) \rightarrow GL(n) \rightarrow 1$$

and the fact that  $X$  is affine, we conclude that  $H^1(X, G_a^n) = 0$ , and therefore that  $\tilde{X}/X$  is a vector bundle. So it has a section.  $\square$

**Proposition 4.3.8.**

- (1) Every equidimensional quasiprojective morphism possesses a cellular quasi-filtration with respect to a given controlled sheaf  $\mathcal{F}$ .
- (2) Every morphism of equidimensional quasiprojective schemes over  $S$  can be lifted to a morphism of cellular quasi-filtered varieties with respect to a controlled constructible sheaf on the target.
- (3) The category of cellular quasi-filtrations of a fixed pair  $(X, \mathcal{F})$  is connected i.e. any two objects can be connected by a chain of morphisms.

*Proof.* The first two statements follow immediately from lemmas 4.3.4 and 4.3.6. The remaining part, take a bit more work.

First we treat the special case of (3) for cellular filtrations. Given two such filtrations  $X_\bullet, X'_\bullet$ , lemma 4.3.4 shows that there is third cellular filtration  $X''_\bullet \supseteq X_\bullet \cup X'_\bullet$ . Now we prove it general. Suppose that we have cellular quasi-filtrations  $(S^\beta, \tilde{X} \rightarrow X, \tilde{X}^\beta)$  and  $(T^\gamma, \tilde{Y} \rightarrow X, \tilde{Y}^\gamma)$ . Take the fibre product  $\tilde{Z} = \tilde{X} \times_X \tilde{Y}$ . By lemma 4.3.7,  $\tilde{Z} \rightarrow \tilde{X}$  and  $\tilde{Z} \rightarrow \tilde{Y}$  admit sections  $\sigma$  and  $\tau$ . Lemma 4.3.4 shows that we can refine  $\sigma(\tilde{X}^\beta) \cup \tau(\tilde{Y}^\gamma)$  to a cellular filtration  $\tilde{Z}^\bullet$  of  $Z$ . By refining  $\tilde{X}^\bullet$  and  $\tilde{Y}^\bullet$ , we obtain a diagram of cellular quasi-filtrations

$$(\tilde{X}, \tilde{X}^\bullet) \rightarrow (\tilde{X}, \tilde{X}'^\bullet) \leftarrow (\tilde{Z}, \tilde{Z}^\bullet) \rightarrow (\tilde{Y}', \tilde{Y}'^\bullet) \leftarrow (\tilde{Y}, \tilde{Y}^\bullet)$$

□

Given a filtration  $X_\bullet \subset X$  by closed sets and a sheaf  $\mathcal{F}$ , we have a spectral sequence

$$E_1^{pq} = H_S^{p+q}(X_p, X_{p-1}; \mathcal{F}) \Rightarrow H_S^{p+q}(X, \mathcal{F})$$

cf [Ar, (10)]. When this is cellular, this reduces to an isomorphism at  $E_2$ . Then putting this remark together with the above results yields

**Lemma 4.3.9.** *Suppose that  $(\pi : \tilde{X} \rightarrow X, \tilde{X}_\bullet)$  is cellular with respect to  $j_{XY!}F$ .  $H_S^i(X, Y; F)$  is isomorphic to the  $i$ th cohomology of the complex*

$$\dots H_S^i(\tilde{X}_i, (\pi^{-1}Y \cap \tilde{X}_i) \cup \tilde{X}_{i-1}) \rightarrow H_S^{i+1}(\tilde{X}_{i+1}, (\pi^{-1}Y \cap \tilde{X}_{i+1}) \cup \tilde{X}_i) \dots$$

## 4.4 Tensor products

We have a product structure on  $\Delta(S, \Sigma)$  (and  $\Delta_{eq}(S, \Sigma)$ ) given by

$$(X \rightarrow S, Y, i, w) \times (X' \rightarrow S, Y', i', w') = (X \times_S X' \rightarrow S, X \times_S Y' \cup X' \times_S Y, i+i', w+w')$$

which makes it into a monoid in the category of graphs with unit  $(id_S, \emptyset, 0, 0)$ . Unfortunately, this does not immediately lead to a product on  $\mathcal{M}(S, \Sigma)$ . The problem has to do with the Künneth formula. To remedy this, we define a full subgraph

$$\Delta_{cell}(S, \Sigma) \subset \Delta_{eq}(S, \Sigma)$$

The objects of  $\Delta_{cell}$  consist of quadruples  $(X \rightarrow S, Y, i, w)$  such that  $X \rightarrow S$  is affine and such that  $H_S^j(X, Y) = 0$  unless  $j = i$ , and such that  $X - Y \rightarrow S$  is smooth. Thanks to Künneth's formula, we have a commutative diagram

$$\begin{array}{ccc} \Delta_{cell}(S, \{S\}) \times \Delta_{cell}(S, \Sigma) & \longrightarrow & \Delta_{cell}(S, \Sigma) \\ \downarrow H_{\{S\}} \times H_{\Sigma} & & \downarrow H_{\Sigma} \\ F\text{-mod} \times F\text{-mod} & \xrightarrow{\otimes} & F\text{-mod} \end{array}$$

leading to a product

$$End^{\vee}(H|_{\Delta_{cell}(S, \{S\})})\text{-comod} \times End^{\vee}(H|_{\Delta_{cell}(S, \Sigma)})\text{-comod} \rightarrow End^{\vee}(H|_{\Delta_{cell}(S, \Sigma)})\text{-comod}$$

With this,  $\mathcal{PM}_{cell}(S) = End^{\vee}(H|_{\Delta_{cell}(S, \{S\})})\text{-comod}$  becomes a tensor category. We form the associated stack  $\mathcal{M}_{cell}(S, \Sigma)$  as before. The tensor product extends to this. To summarize

**Lemma 4.4.1.** *There are tensor products*

$$\mathcal{M}_{cell}(S, \{S\}) \times \mathcal{M}_{cell}(S, \Sigma) \rightarrow \mathcal{M}_{cell}(S, \Sigma)$$

*compatible, via the forgetful functor  $U$ , with the vector space tensor product. With this structure  $\mathcal{M}_{cell}(S, \{S\})$  becomes a tensor category.*

The key point is:

**Theorem 4.4.2.** *The category  $\mathcal{M}_{cell}(S)$  is equivalent to  $\mathcal{M}_{eq}(S)$ .*

Before giving the proof, we give a construction. Let  $C^{[0, \infty)}(\mathcal{M}(S, \Sigma))$  be the category of bounded complexes supported in nonnegative degrees. Let  $\mathcal{H}^i : C^{[0, \infty)}(\mathcal{M}(S, \Sigma)) \rightarrow \mathcal{M}(S, \Sigma)$  denote the  $i$ th cohomology functor. Then composition gives a functor  $R_B \circ \mathcal{H}^*$  from  $C^{[0, \infty)}(\mathcal{M}(S, \Sigma))$  to the category  $GrSh(S((\mathbb{C})))$  of  $[0, \infty)$ -graded sheaves. Let  $\mathcal{C}(S, \Sigma)$  be the so called comma category whose objects are triples

$$(K^{\bullet}, M, \phi : R_B(M) \rightarrow R_B \circ \mathcal{H}^0(K^{\bullet}))$$

where  $K^{\bullet} \in ObC^{[0, \infty)}(\mathcal{M}(S, \Sigma))$  and  $M \in Ob\mathcal{M}(S, \Sigma)$ . Morphisms are pairs  $K_1^{\bullet} \rightarrow K_2^{\bullet}$ ,  $M_1 \rightarrow M_2$  satisfying obvious compatibilities. Let  $\mathcal{C}_{iso}(S, \Sigma)$  be the full subcategory consisting of triples for which  $\phi$  is an isomorphism. We can identify  $F^2\text{-mod}$  with  $F\text{-mod} \times F\text{-mod}$ . There is a faithful exact functor  $U_2 : \mathcal{C}(S, \Sigma) \rightarrow (F\text{-mod})^2$  given by  $(K^{\bullet}, M, \phi) \mapsto (\prod_i U(K^i) \times U(M)$ .

Given a simple quasi-filtration  $(T \rightarrow S, T_{\bullet})$  and a stratification  $\Sigma$ , choose base points  $s$  for  $(S, \Sigma)$  and  $t_{\bullet} \in T_{\bullet}$ . We define a functor  $H^{\#} : \Delta(S, \Sigma)^{op} \rightarrow \mathcal{C}(S, \Sigma)_{iso}$  as follows. On objects

$$H^{\#}(X \rightarrow S, Y, i, w) =$$

$$((h_S^0(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \rightarrow h_S^1(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \rightarrow \dots)[i]; h_S^i(X, Y); \phi) \quad (9)$$

where the differentials of the complex are connecting maps and  $\phi$  is given by lemma 4.3.9.



**Definition 4.4.3.**  $\mathcal{P}\mathcal{M}^\#(f, T \rightarrow S, T_\bullet, \Sigma) = \text{End}_{F^2}^\vee(H^\# \circ U_2)\text{-comod.}$

This carries an exact faithful embedding into  $F^2\text{-mod.}$  The categories  $\mathcal{P}\mathcal{M}^\#(f, T \rightarrow S, T_\bullet, \Sigma)$  fibred over  $S$ -schemes. We can form the associated stack  $\mathcal{M}^\#(f, T \rightarrow ST_\bullet, \Sigma)$  for the Zariski topology. This can be constructed explicitly by following the procedure outlined at the end of § 4.1. In order to simplify notation, we usually just write this as  $\mathcal{M}^\#(T_\bullet)$ , when the rest of the data is understood. We let  $h_{T_\bullet}(X, Y)(w)$  denote the object of this category associated to  $(X, Y, i, w)$ . We have a functor  $\mathcal{M}^\#(T_\bullet) \rightarrow \mathcal{C}(S, \Sigma)_{\text{iso}}$ , and a functor  $p : \mathcal{M}^\#(T_\bullet) \rightarrow \mathcal{M}(S, \Sigma)$  given as a composition of this with the functor  $\mathcal{C}(S, \Sigma)_{\text{iso}} \rightarrow \mathcal{M}(S, \Sigma)$  given by projection onto the second factor. From lemma 4.1.6, we obtain

**Lemma 4.4.4.**  $\mathcal{M}(S, \Sigma)$  is equivalent to  $\mathcal{M}^\#(T_\bullet)/\ker p$ .

*Proof of theorem 4.4.2.* Restriction gives a functor  $\iota : \mathcal{P}\mathcal{M}_{\text{cell}}(S, \Sigma) \rightarrow \mathcal{P}\mathcal{M}_{\text{eq}(S, \Sigma)}$  which is necessarily exact and faithful. It suffices to show that this is an equivalence, because it will then induce an equivalence of the corresponding stacks  $\mathcal{M}_{\text{cell}}(S, \Sigma) \sim \mathcal{M}_{\text{eq}(S, \Sigma)}$ . We show that  $\iota$  is essentially surjective and full, and for this it suffices to have a right inverse up to natural equivalence. This is induced by the functor  $\mathcal{P}\mathcal{M}^\#(T_\bullet) \rightarrow \mathcal{M}_{\text{cell}}(S)$  given by

$$(X \rightarrow S, Y, i, w) \mapsto \mathcal{H}^0(h_S^0(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \rightarrow h_S^1(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \rightarrow \dots)[i]$$

□

**Corollary 4.4.5.** *There is a Künneth decomposition for motives associated to objects in  $\Delta_{\text{eq}}(S, \{S\})$ :*

$$h_S^i(X \times_S X', X \times_S Y' \cup X' \times_S Y) \cong \bigoplus_{j+j'=i} h_S^j(X, Y) \otimes h_S^{j'}(X', Y')$$

*Proof.* This follows from the theorem and lemma 4.4.1. □

For objects in  $\Delta_{\text{eq}}(S, \{S\})$ , we get exterior products

$$h_S^j(X, Y) \otimes h_S^{j'}(X', Y') \rightarrow h_S^{j+j'}(X \times_S X', X \times_S Y' \cup X' \times_S Y)$$

and cup products

$$h_S^j(X, Y) \otimes h_S^{j'}(X, Y) \rightarrow h_S^{j+j'}(X, Y)$$

by composing this with the restriction to the diagonal. Corollary 2.4.1 shows that these products are compatible with the standard tensor products on the categories of classical and étale local systems.

## 4.5 Ind objects

Let  $\text{Ind-}\mathcal{A}$  denote the category of Ind-objects of a category  $\mathcal{A}$  obtained by formally adjoining filtered colimits [KS]. This is abelian, when  $\mathcal{A}$  is [loc. cit.]. This can be given a concrete description in many cases. For example, it is well known that  $\text{Ind-}F\text{-mod}$  can be identified with  $F\text{-Mod}$ . We extend this to sheaves. Recall that an object  $c$  of an additive category is compact or finitely presented if  $\text{Hom}(c, -)$  commutes with arbitrary small coproducts. For example, a finite dimensional vector space  $V$  is seen to be compact in the category of all vector spaces, because an element of  $\text{Hom}(V, -)$  is determined by its value on a finite basis. For essentially the same reasons, we have:

**Lemma 4.5.1.** *A constructible sheaf is compact in the category  $\text{Sh}(S)$  of sheaves of  $F$ -modules.*

*Proof.* For any collection of sheaves  $\mathcal{F}, \mathcal{G}_i$ , we have canonical map

$$\kappa : \bigoplus_{i \in I} \text{Hom}(\mathcal{F}, \mathcal{G}_i) \rightarrow \text{Hom}(\mathcal{F}, \bigoplus_{i \in I} \mathcal{G}_i)$$

Injectivity can be checked on stalks. Consider the projection

$$p_j : \bigoplus \mathcal{G}_i \rightarrow \prod \mathcal{G}_i \rightarrow \mathcal{G}_j$$

One checks that  $\phi \in \text{Hom}(\mathcal{F}, \bigoplus \mathcal{G}_i)$  lies in  $\text{im}(\kappa)$  precisely when it has finite support in the sense that there exists a finite subset  $J \subset I$  such that the germs  $p_i(\phi_s) = 0$  for all  $s \in S$  and  $i \notin J$ .

Suppose  $\mathcal{F}$  is constructible with respect to a necessarily finite Zariski stratification  $\Sigma$ . Let  $\pi_\sigma : \tilde{\sigma} \rightarrow \sigma$  denote the universal cover of a stratum. Choose bases of cardinality say  $n_\sigma$  for each  $H^0(\pi_\sigma^* \mathcal{F})$ . Then  $\phi \in \text{Hom}(\mathcal{F}, \bigoplus \mathcal{G}_i)$  is determined by its image

$$r(\phi) = (\pi_\sigma^* \phi) \in \prod_{\sigma} H^0(\pi_\sigma^* \text{Hom}(\mathcal{F}, \bigoplus \mathcal{G}_i)),$$

i.e. the map  $r$  is injective. In explicit terms,  $r(\phi)$  is given by a collection of  $n_\sigma$  sections of  $\bigoplus \pi_\sigma^* \mathcal{G}_i$  for each  $\sigma$ . The projections  $r(p_j(\phi))$  are given by simply projecting these sections to  $\mathcal{G}_j$ . It should now be clear that  $\phi$  has finite support.  $\square$

**Corollary 4.5.2.** *There is a fully faithful exact embedding of  $\text{Ind-Cons}(S)$  into  $\text{Sh}(S)$ .*

*Proof.* This follows from [KS, prop 6.3.4] and the exactness of filtered colimits.  $\square$

Therefore we have an exact faithful functor

$$\text{Ind-}\mathcal{M}(S) \rightarrow \text{Ind-Cons}(S) \rightarrow \text{Sh}(S)$$

given by composition. This is also denoted by  $R_B$ .

## 5 Direct Images

### 5.1 Direct Images (abstract construction)

We start by giving a general construction of direct images. Set  $DM(S) = D(\text{Ind-}\mathcal{M}(S))$ . Fix a morphism  $f : S \rightarrow Q$ . Since the functor  $f^*$  is exact, it extends to an exact functor on  $\text{Ind-}\mathcal{M}(Q) \rightarrow \text{Ind-}\mathcal{M}(S)$ . Thus we have an extension  $f^* : DM(Q) \rightarrow DM(S)$  as a triangulated functor.

**Theorem 5.1.1.** *If  $f : S \rightarrow Q$  is a morphism of quasiprojective varieties, then there is a triangulated functor  $rf_* : DM(S) \rightarrow DM(Q)$  which is right adjoint to  $f^*$ .*

*Proof.* The theorem will be deduced from a form of Brown's representability theorem due to Franke [F]. Note that the extension  $f^*$  to  $\text{Ind-}\mathcal{M}(-)$  commutes with filtered direct limits and therefore coproducts. Since  $\text{Ind-}\mathcal{M}(S)$  is a Grothendieck category by [KS, thm 8..6.5], Franke's theorem [F, thm 3.1] implies that

$$M \mapsto \text{Hom}(f^*M, N)$$

is representable by an object  $rf_*N$ . The map  $N \rightarrow rf_*N$  extends to a functor which is necessarily the right adjoint, cf. [N, p223]. Moreover, this is automatically triangulated by [Li2, prop 3.3.8].  $\square$

The abstract construction is not terribly useful by itself. We would really like more:

**Definition 5.1.2.** *Let us say that a morphism  $f : S \rightarrow Q$  possesses a good direct image if*

1.  $rf_*(D^b(\mathcal{M}(Q))) \subseteq D^b(\mathcal{M}(S))$ , where we identify  $D^b\mathcal{M}$  with a triangulated subcategory of  $DM$ .
2. For each  $M \in \mathcal{M}(S)$ . The map

$$R_B rf_* M \rightarrow \mathbb{R}f_* R_B M$$

adjoint to the canonical map

$$f^* R_B rf_* M \cong R_B f^* rf_* M \rightarrow R_B M$$

is an isomorphism.

The following lemma gives a criterion for checking this.

**Lemma 5.1.3.** *Suppose that  $r'f_* : D^b(\mathcal{M}(S)) \rightarrow D^b(\mathcal{M}(Q))$  is a functor equipped with natural transformations*

$$\eta : 1 \rightarrow r'f_* f^*$$

and

$$\epsilon : f^* r'f_* \rightarrow 1$$

such that:

(1) The map

$$R_B r' f_* M \rightarrow \mathbb{R} f_* R_B M$$

adjoint to

$$R_B \epsilon : f^* R_B r' f_* M \rightarrow R_B M$$

is an isomorphism.

(2) The composition

$$R_B M \xrightarrow{R_B \epsilon} R_B r' f_* f^* M \xrightarrow{(1)} \mathbb{R} f_* f^* R_B M$$

coincides with the adjunction map  $1 \rightarrow \mathbb{R} f_* f^*$ .

Then  $r' f_*$  is right adjoint to  $f^*$ . So, in particular,  $f$  has a good direct image.

*Proof.* It is enough to check that the compositions

$$f^* \rightarrow f^* r' f_* f^* \rightarrow f^*$$

and

$$r' f_* \rightarrow r' f_* f^* r' f_* \rightarrow r' f_*$$

are both identity [M, chap IV]. Since the realizations  $R_B$  are embeddings, this follows from the compatibility of  $\eta, \epsilon$  with the usual adjunctions on the categories of sheaves.  $\square$

As a prelude to a more general result proved later, we show that  $f : S \rightarrow Q$  has a good direct image when it is a closed immersion. By 2.2.10, the map  $\Delta(S, \Sigma) \rightarrow \mathcal{M}(Q)$  given by

$$(X \rightarrow S, Y, i, w) \mapsto h_Q^i(X, Y)(w)$$

induces an exact functor  $f_* : \mathcal{M}(S) \rightarrow \mathcal{M}(Q)$ .

**Proposition 5.1.4.** *If  $f$  is a closed immersion, then  $f_*$  satisfies the conditions of lemma 5.1.3. Therefore,  $f_*$  is right adjoint to  $f^*$ .*

*Proof.* We have to construct natural transformations  $\eta : 1 \rightarrow f_* f^*$  and  $\epsilon : f^* f_* \rightarrow 1$  satisfying the conditions of lemma 5.1.3.

We can see from the construction that  $f^* f_* h_S^i(X, Y)(w)$  is equal to  $h_S^i(X, Y)(w)$ . Thus we have a canonical isomorphism, which gives the required map  $\epsilon$ . This clearly satisfies lemma 5.1.3 (1).

Let  $Mor\mathcal{M}(Q)$  denote the category whose objects are morphisms of  $\mathcal{M}(Q)$ , and whose morphisms are commutative squares. Let  $Mor' \subset Mor(\mathcal{M}(Q))$  denote the subcategory of morphisms  $M_1 \rightarrow M_2$  such that there is a commutative diagram

$$\begin{array}{ccc} R_B M_1 & \longrightarrow & f_* f^* R_B M_1 \\ \downarrow = & & \downarrow \cong \\ R_B M_1 & \longrightarrow & R_B M_2 \end{array}$$

commutes. The morphisms are squares

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow h_1 & & \downarrow h_2 \\ M'_1 & \longrightarrow & M'_2 \end{array}$$

such that  $R_B h_2$  is given by  $f_* f^* h_1$ . The functor  $(M_1 \rightarrow M_2) \mapsto M_1$  is clearly faithful and exact. Therefore by corollary 2.2.10, we get a functor  $\mathcal{M}(Q) \rightarrow \mathcal{M}or'$  such that

$$h_Q^i(X, Y)(w) \mapsto [h_Q^i(X, Y)(w) \rightarrow h_Q^i(X_S, Y_S)(w)]$$

This gives the canonical adjunction  $\eta : 1 \rightarrow f_* f^*$ . □

Combining this with proposition 4.2.5 yields:

**Lemma 5.1.5.** *Let  $j : S \rightarrow \bar{S}$  be an open immersion with complement  $i : \bar{S} - S \rightarrow S$ . Then for any  $\mathcal{F} \in \mathcal{M}(\bar{S})$ , there is a canonical exact sequence*

$$0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$$

where  $i : \partial \bar{S} \rightarrow \bar{S}$  is the inclusion.

## 5.2 Direct Images (conclusion)

We come to the main technical result of this paper.

**Theorem 5.2.1.** *A morphism  $f : S \rightarrow Q$  possesses a good direct image if either  $f$  is projective or  $Q$  is a point.*

The proof, which will be broken into a series of lemmas, is quite messy, although the basic idea is rather simple. The hypothesis of the theorem is used in the following way: a controlled pair  $(g : X \rightarrow S, Y)$  determines a controlled pair  $(f \circ g : X \rightarrow Q, Y)$  when either  $f$  is projective or trivially when  $Q$  is a point. Let us say that  $(X \rightarrow S, Y, i, 0) \in \Delta(S)$  is a  $f$ -cellular if  $H_Q^j(H_S^i(X, Y))$  is zero for all but one value of  $j$ , say  $j = m$ . Then the proof will show that  $h_Q^{m+i}(X, Y)[m]$  will give a model for  $rf_*(h_S^i(X, Y))$ . This clearly maps to  $\mathbb{R}f_* H_S^i(X, Y)$  under  $R_B$ , so “goodness” is verified in this case. In general, we will realize  $rf_* M$  as an explicit complex of  $f$ -cellular motives, which maps to  $\mathbb{R}f_* R_B M$ . Since this construction depends on auxiliary choices, it is necessary work on a bigger category  $\mathcal{M}^\#$ , lying over  $\mathcal{M}$ , in order to get a functor temporarily called  $q$ . The final step is to show that  $q$  descends to a functor on the derived categories, and that this is indeed the adjoint to  $f^*$ .

By factoring  $f$  through a closed immersion followed by a projection, and applying proposition 5.1.4, we can see that to prove theorem 5.2.1, we can assume that  $f : S \rightarrow Q$  is flat and therefore equidimensional. Let  $g : X \rightarrow S$  be a quasi-projective morphism with  $Y \subset X$  closed, such that  $(X \rightarrow S, Y)$

is controlled. By proposition 4.3.8, we can find a quasi-filtration  $(\{Q^\bullet\}, T \rightarrow S, T_\bullet)$  which is cellular with respect to the sheaves  $H_S^*(X, Y)$ . To simplify the discussion, let us suppose that this is simple, i.e. that the cover  $\{Q^\bullet\} = \{Q\}$ . Consider the commutative diagram

$$\begin{array}{ccccccc} X_{T_\bullet} & \longrightarrow & X_T & \longrightarrow & X & & \\ \downarrow & & \downarrow & & \downarrow g & \searrow f \circ g & \\ T_\bullet & \longrightarrow & T & \longrightarrow & S & \xrightarrow{f} & Q \end{array}$$

where the squares are cartesian. Then we can form a complex of sheaves

$$\mathcal{K}_i^\bullet = H_Q^i(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \rightarrow H_Q^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \rightarrow \dots$$

where the differentials are the connecting maps.

**Proposition 5.2.2.** *With the previous assumptions, there is a canonical isomorphism*

$$H_Q^j(S, H_S^i(X, Y)) \cong \mathcal{H}^j(\mathcal{K}_i^\bullet)$$

where  $\mathcal{H}^j$  stands for the  $j$ th cohomology sheaf.

*Proof.* When  $Q$  is a point and  $Y = \emptyset$ , this was originally proved in [Ar, thm 3.1]. The general case can be proved by the same method. However, a slightly cleaner alternative is to deduce it from [CM]. Since the model case ( $Q = pt, Y = \emptyset$ ) is spelled out in detail in [C], we will be content to give the broad outline. Since both sides of the purported isomorphism are stable under base change to  $T$ , we can assume that  $T = S$ . Let  $L = \mathbb{R}f_* \mathbb{R}g_* j_{XY}! F$ . We consider two filtrations on  $L$ . The first is defined by truncations  $P^\bullet(L) = \mathbb{R}f_* \tau_{\leq -\bullet} \mathbb{R}g_* j_{XY}! F$ , so that  $Gr_P^\bullet L = \mathbb{R}f_* H_S^{-\bullet}(X, Y)$ . The second  $\mathcal{F}$  is the filtration on  $L$  associated to  $T_\bullet$ ,

$$\mathcal{F}^\bullet L = \mathbb{R}f_* j_{ST_\bullet}! j_{ST_\bullet}^* \mathbb{R}g_* j_{XY}! F$$

Then

$$Gr_{\mathcal{F}}^\bullet L \cong \mathbb{R}f_* j_{T_\bullet T_{\bullet-1}}! j_{T_\bullet T_{\bullet-1}}^* \mathbb{R}g_* j_{XY}! F \cong \mathbb{R}f_* \mathbb{R}g_* j_{X_{T_\bullet}, Y_{T_\bullet} \cup X_{T_{\bullet-1}}}! F$$

holds. Then the cellularity of  $T_\bullet$  implies that the assumptions of [CM, prop 5.6.1] are satisfied. Therefore we have a natural filtered quasi-isomorphism

$$(L, P) \cong (L, Dec(\mathcal{F}))$$

where  $Dec(\mathcal{F})$  is the shifted filtration associated to  $\mathcal{F}$ . This has the property that  $E_1(Dec(\mathcal{F})) = E_2(\mathcal{F})$  (c.f. [De1]). Thus it follows that there is an isomorphism of the spectral sequences associated to  $P$  and  $Dec(\mathcal{F})$ . The spectral sequence for  $P$  is Leray with a shift in indices, and in particular  $E_1(P) = H_Q^j(S, H_S^i(X, Y))$ . On the other hand, we can identify

$$E_1(\mathcal{F}) = H_Q^\bullet(X_{T_\bullet}, Y_{T_\bullet} \cup X_{T_{\bullet-1}}),$$

using the fact that  $H_Q^\bullet(X_{T_\bullet}, \dots)$  commutes with base change because the maps are controlled. Hence  $E_1(Dec(\mathcal{F})) = E_2(\mathcal{F}) = \mathcal{H}^j(\mathcal{K}_i)$ .  $\square$

We now construct an auxiliary category  $\mathcal{M}^\#$  by a variation of the method used in section 4.4. A pair of stratifications  $\Sigma$  of  $S$  and  $\Lambda$  of  $Q$  will be called  $f$ -admissible if each  $\sigma \in \Sigma$  is a fibre bundle over some  $\lambda \in \Lambda$ . Given a stratification  $\Sigma'$  of  $S$ , there exists an admissible pair  $(\Sigma, \Lambda)$  for which  $\Sigma$  refines  $\Sigma'$ . Any pair of stratifications can always be refined so that admissibility holds. Choose admissible stratifications  $\Sigma$  and  $\Lambda$ . Then composition gives a functor  $R_B \circ \mathcal{H}^*$  from  $C^{(0, \infty)}(\mathcal{M}(Q, \Lambda))$  to the category  $GrSh(Q(\mathbb{C}))$  of  $[0, \infty)$ -graded sheaves. We also have a functor  $H_Q^* \circ R_B : \mathcal{M}(S, \Sigma) \rightarrow GrSh(Q(\mathbb{C}))$ . Let  $\mathcal{C}(S, \Sigma)$  be the so called comma category whose objects are triples

$$(K^\bullet, M, \phi : H_Q^* \circ R_B(M) \rightarrow R_B \circ \mathcal{H}^*(K^\bullet))$$

where  $K^\bullet \in ObC^{(0, \infty)}(\mathcal{M}(Q, \Lambda))$  and  $M \in Ob\mathcal{M}(S, \Sigma)$ . Morphisms are pairs  $K_1^\bullet \rightarrow K_2^\bullet, M_1 \rightarrow M_2$  satisfying obvious compatibilities. Let  $\mathcal{C}_{iso}(S, \Sigma)$  be the full subcategory consisting of triples for which  $\phi$  is an isomorphism. We can identify  $F^2\text{-mod}$  with  $F\text{-mod} \times F\text{-mod}$ . There is a faithful exact functor  $U_2 : \mathcal{C}(S, \Sigma) \rightarrow (F\text{-mod})^2$  given by  $(K^\bullet, M, \phi) \mapsto (\prod_i U(K^i) \times U(M)$ .

Given a simple cellular quasi-filtration  $(T \rightarrow S, T_\bullet)$  and a stratification  $\Sigma$ , choose base points  $s$  for  $(S, \Sigma)$  and  $t_\bullet \in T_\bullet$ . We define a functor  $H^\# : \Delta(S, \Sigma)^{op} \rightarrow \mathcal{C}(S, \Sigma)_{iso}$  as follows. On objects

$$H^\#(X \rightarrow S, Y, i, w) = (h_Q^i(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}})(w) \rightarrow h_Q^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}})(w) \rightarrow \dots; h_S^i(X, Y)(w); \phi) \quad (10)$$

where the differentials of the complex are connecting maps and  $\phi$  is given by proposition 5.2.2. We can extend this to nonsimple cellular quasi-filtrations  $(\{Q^\bullet\}, T \rightarrow S, T_\bullet)$  as follows. For notational simplicity, we assume that the cover consists of two sets  $\{Q^0, Q^1\}$  and that  $T_\bullet^0, T_\bullet^1$  can be refined to  $T_\bullet^{01}$  on  $Q^{01} = Q^0 \cap Q^1$ . Let  $j_0, j_1, j_{01}$  denote the inclusions of  $Q^0, Q^1$  and  $Q^{01}$  into  $Q$  respectively. Since the varieties  $T_\bullet^0$  can be extended over  $Q$  by taking closures, the extensions by zero

$$M_\alpha^j = j_{\alpha!} h_{Q^\alpha}^{i+j}(X_{T_j^\alpha}, Y_{T_j^\alpha} \cup X_{T_{j-1}^\alpha})(w)$$

are defined. We can now define  $H^\#(X \rightarrow S, Y, i, w)$  by taking  $h_S^i(X, Y)(w)$  as the second component as above. For the first component, we use the complex

$$\ker[M_0^\bullet \oplus M_1^\bullet \rightarrow M_{01}^\bullet]$$

where the map is given the difference of restrictions. This complex is quasi-isomorphic to  $M_\alpha^\bullet$  on  $Q^\alpha$ . The quasi-isomorphism  $\phi$  can be thus extended, so that  $H^\#(X \rightarrow S, Y, i, w) \in \mathcal{C}_{iso}$

We let  $\mathcal{PM}^\#(f, \{Q^\bullet\}, T \rightarrow S, T_\bullet, \Sigma) = \text{End}_{F_2}^\vee(H^\# \circ U_2)\text{-comod}$ , and let  $\mathcal{M}^\#(f, \{Q^\bullet\}, T \rightarrow S, T_\bullet, \Sigma)$  denote the associated stack. This carries an exact faithful embedding into  $F^2\text{-mod}$ . In order to simplify notation, we usually just write these as  $\mathcal{PM}^\#(T_\bullet)$  and  $\mathcal{M}^\#(T_\bullet)$ . We let  $h_{T_\bullet}(X, Y)(w)$  denote the object of this category associated to  $(X, Y, i, w)$ . We have a functor

$\mathcal{M}^\#(T_\bullet) \rightarrow \mathcal{C}(S, \Sigma)_{iso}$ . We can compose this with the projections to get functors  $p : \mathcal{P}\mathcal{M}^\#(T_\bullet) \rightarrow \mathcal{M}(S, \Sigma)$  and  $q : \mathcal{M}^\#(T_\bullet) \rightarrow C^{[0, \infty)}(\mathcal{M}(Q, \Lambda))$ . These extend to functors on  $\mathcal{M}^\#(T_\bullet)$  denoted by the same symbols. From corollary 2.3.2, we obtain

**Lemma 5.2.3.**  $\mathcal{M}(S, \Sigma)$  is equivalent to  $\mathcal{M}^\#(T_\bullet)/\ker p$ .

Let

$$\bar{q} : \mathcal{M}^\#(T_\bullet) \rightarrow D^b(\mathcal{M}(Q, \Lambda))$$

denote the composition of  $q$  with the canonical map.

**Lemma 5.2.4.** There is a functor  $r'f_*$  fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{M}^\#(T_\bullet) & & \\ \downarrow & \searrow \bar{q} & \\ D^b(\mathcal{M}(S, \Sigma)) & \xrightarrow{r'f_*} & D^b(\mathcal{M}(Q, \Lambda)) \end{array}$$

such that there is a natural isomorphism  $R_B r'f_* \cong \mathbb{R}f_* R_B$ . This functor is independent of the choice of quasi-filtration.

*Proof.* We note that  $q$  and  $\bar{q}$  can be extended to  $C^b(\mathcal{M}^\#(T_\bullet))$  by taking the total complex associated to the double complex induced by these functors. By lemma 5.2.3,  $\mathcal{M}(S)$  is equivalent to  $\mathcal{M}^\#(T_\bullet)/\ker p$ . This extends to an equivalence  $C^b(\mathcal{M}(S)) \sim C^b(\mathcal{M}^\#(T_\bullet)/\ker p)$ . From the definition of  $\mathcal{C}_{iso}(S)$ , it follows that

$$R_B(\mathcal{H}^i(q(M))) \cong H_Q^i(S, R_B(p(M))) \quad (11)$$

as functors in  $M \in C^b(\mathcal{M}^\#(T_\bullet))$ . Since  $R_B$  is faithful, this isomorphism implies that  $\mathcal{H}^i \circ q$  factors through  $D^b(\mathcal{M}^\#(T_\bullet)/\ker p)$ . We can summarize all of this by the commutative diagram

$$\begin{array}{ccc} C^b(\mathcal{M}^\#(T_\bullet)) & \xrightarrow{\bar{q}} & D^b(\mathcal{M}(Q, \Lambda)) \\ \downarrow p & \searrow & \downarrow R_B \\ & D^b(\mathcal{M}^\#(T_\bullet)/\ker p) & \\ & \swarrow h & \\ D^b(\mathcal{M}(S, \Sigma)) & \xrightarrow{\mathbb{R}f_* \circ R_B} & D^b(Sh(Q)) \end{array}$$

$\sim$   $e$

Since  $e$  is an equivalence, it has an inverse. The desired functor  $r'f_*$  would be given by  $h \circ e^{-1}$ , but since it depends, a priori, on the choice of the quasi-filtration, we temporarily denote it by  $r'f_{*, Q, T_\bullet}$ . Given a second cellular quasi-filtration  $(T'_\bullet \rightarrow S, T'_\bullet)$ , we wish to show that there is a canonical isomorphism  $r'f_{*, Q, T_\bullet} \cong r'f_{*, Q, T'_\bullet}$ . By proposition 4.3.8, we can assume that there is a morphism  $(T'_\bullet \rightarrow S, T'_\bullet) \rightarrow (T_\bullet \rightarrow S, T_\bullet)$  in  $QVar_Q$  covering the identity  $id :$



$S \rightarrow S$ . Therefore the complexes defining  $r'f_{*,Q,T_\bullet}$  and  $r'f_{*,Q,T'_\bullet}$  become quasi-isomorphic since the constructions factor through  $\mathcal{C}(id)_{iso}$ . So we can now omit the second subscript.  $\square$

To finish the proof of the theorem, we will verify the conditions of lemma 5.1.3 which entails constructing adjunctions.

**Lemma 5.2.5.** *Given maps  $g : X \rightarrow S$  and  $f : S \rightarrow Q$  of topological spaces, consider a commutative diagram*

$$\begin{array}{ccc} X \times_Q S & \xrightleftharpoons[\Gamma]{\pi} & X \\ \downarrow p & & \downarrow f \circ g \\ S & \xrightarrow{f} & Q \end{array}$$

where  $\Gamma$  is the inclusion of the graph of  $g$ . Then the adjunction map  $f^*\mathbb{R}f_*\mathbb{R}g_* \rightarrow \mathbb{R}p_*\pi^*$  is the composition of the base change map

$$f^*\mathbb{R}(f \circ g)_* \rightarrow \mathbb{R}p_*\pi^*$$

and the adjunction

$$\mathbb{R}p_*\pi^* \rightarrow \mathbb{R}p_*\mathbb{R}\Gamma_*\Gamma^*\pi^* = \mathbb{R}g_*$$

*Proof.* This follows by applying [Li2, prop 3.7.2ii] to the diagram

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow \Gamma & & \downarrow id \\ X \times_Q S & \xrightarrow{\pi} & X \\ \downarrow p & & \downarrow f \circ g \\ S & \xrightarrow{f} & Q \end{array}$$

$\square$

**Corollary 5.2.6.** *If the first base change map is an isomorphism,  $f^*\mathbb{R}f_*\mathbb{R}g_* \rightarrow \mathbb{R}g_*$  can be identified with the map  $\mathbb{R}p_*\pi^* \rightarrow \mathbb{R}g_*$  induced by  $\Gamma$ .*

**Lemma 5.2.7.** *There is a morphism  $\epsilon : f^*r'f_* \rightarrow 1$  compatible with the adjunction  $\epsilon : f^*\mathbb{R}f_* \rightarrow 1$ .*

*Proof.* The previous corollary implies that

$$\Gamma^* : H_S^i(X \times_Q S, Y \times_Q S) \rightarrow H_S^i(X, Y)$$

is precisely the adjunction map  $\epsilon$ . We have to lift this to a morphism  $\epsilon : f^*r'f_* \rightarrow 1$ . Let  $\Sigma$  and  $\Lambda$  be an  $f$ -admissible pair of stratifications. Choose a quasi-filtration  $T_\bullet \rightarrow S$ . Define a functor

$$H^\blacklozenge : \Delta(S, \Sigma) \rightarrow F\text{-mod}^3$$

by sending  $(X \rightarrow S, Y, i, w)$  to the direct sum of the three vertices of the diagram

$$\begin{array}{ccc} \mathcal{H}^0(H_S^i(X_{T_0} \times_Q S, Y_{T_0} \times_Q S \cup X_{T_{0-1}} \times_Q S) \rightarrow \dots) & \xleftarrow{\sim} & H_S^i(X \times_Q S, Y \times_Q S) \\ & & \downarrow \Gamma^* \\ & & H_S^i(X, Y) \end{array}$$

We build a category  $\mathcal{M}^\clubsuit(S) = \text{End}F^{3^\vee}(H^\clubsuit)\text{-comod}$ . From the universal property, we have a functor  $h^\clubsuit$  which assigns to  $(X \rightarrow S, Y, i, w)$  the diagram of motives

$$\begin{array}{ccc} \mathcal{H}^0(h_S^i(X_{T_0} \times_Q S, Y_{T_0} \times_Q S \cup X_{T_{0-1}} \times_Q S) \rightarrow \dots) & \xleftarrow{\sim} & h_S^i(X \times_Q S, Y \times_Q S) \\ & & \downarrow \Gamma^* \\ & & h_S^i(X, Y) \end{array}$$

In particular, we obtain a projection  $\mathcal{M}^\clubsuit(S) \rightarrow \mathcal{M}(S)$ . As above one can argue that  $\mathcal{M}^\clubsuit(S)/\ker(p) \sim \mathcal{M}(S)$ . We can see that  $h^\clubsuit$  factors through  $\ker p$ . This determines a functor  $\epsilon : \mathcal{M}(S) \rightarrow \text{Mor}\mathcal{M}(S)$  compatible with adjunction.  $\square$

**Lemma 5.2.8.** *There is a morphism  $\eta : 1 \rightarrow r'f_*f^*$  compatible with the adjunction  $\eta : 1 \rightarrow \mathbb{R}f_*f^*$*

*Proof.* The strategy is similar to the previous argument. Let  $\Sigma$  and  $\Lambda$  and  $T_\bullet \rightarrow S$  be as in the above argument. Define

$$H^\clubsuit : \Delta(Q, \Lambda) \rightarrow F\text{-mod}^2$$

by sending  $(X \rightarrow Q, Y, i, w)$  to the sum of vertices of the diagram

$$\begin{array}{c} H_Q^i(X, Y) \\ \downarrow \\ H_Q^i(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \longrightarrow H_Q^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \longrightarrow \dots \end{array}$$

partitioned so that  $H_Q^i(X, Y)$  corresponds to the first component. Let  $\mathcal{M}^\clubsuit(S) = \text{End}_{F^3}^\vee(H^\clubsuit)\text{-comod}$ . As above, we have a functor  $h^\clubsuit$  sending  $(X \rightarrow Q, Y, i, w)$  to

$$\begin{array}{c} h_Q^i(X, Y) \\ \downarrow \\ h_Q^i(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \longrightarrow h_Q^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \longrightarrow \dots \end{array}$$

This yields the map  $\eta$  once we observe that  $\mathcal{M}^\clubsuit(S)/\ker(p) \sim \mathcal{M}(S)$  where  $p$  is the natural projection.  $\square$

*Proof of theorem.* To finish the proof of the theorem, it is enough to observe that by the previous lemmas, we can apply lemma 5.1.3 to conclude that  $rf_* = r'f'_*$ .  $\square$

### 5.3 Direct image with compact support

**Theorem 5.3.1.** *If  $f : S \rightarrow Q$  is an morphism of quasiprojective varieties, then there is a functor  $h_{c,Q}^i = r^i f_! : \mathcal{M}_c(S; F) \rightarrow \mathcal{M}(Q; F)$ , such that  $R_B(r^i f_!(M)) \cong R^i f_!(S, R_B(M))$ . If  $g : S' \rightarrow S$  is a morphism, there is an isomorphism  $g^* r^i f_!(M) \cong r^i f_!(g^* M)$  compatible with the base change isomorphism on realizations.*

*Proof.* Choose a relative compactification

$$\begin{array}{ccc} S & \xrightarrow{j} & \bar{S} \\ \downarrow f & \searrow \bar{f} & \\ Q & & \end{array}$$

Then we can define  $r^i f_! = (r \bar{f}_*^i) \circ (j_!)$ , provided that we can show that it is well defined. First observe that we have

$$R_B(r \bar{f}_*^i \circ j_! M) = R \bar{f}_*^i \circ j_! R_B(M) = R^i f_! R_B(M)$$

as required. To see that it is independent of the choice, first observe that given a second compactification  $S \rightarrow \tilde{S}$ , we can find a third compactification  $\bar{\bar{S}}$  which dominates both  $\bar{S}$  and  $\tilde{S}$ . For example, we can take  $\bar{\bar{S}}$  equal the closure of diagonal in  $\bar{S} \times_Q \tilde{S}$ . Thus we can assume that  $\bar{\bar{S}} = \tilde{S}$ , and so we can assume that we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{j} & \bar{S} \\ \downarrow f & \searrow \bar{f} & \uparrow \pi \\ Q & & \tilde{S} \\ & \nearrow \tilde{f} & \\ & & \end{array}$$

Therefore, we get a morphism

$$r^i \bar{f}_* j_! M \rightarrow r^i \tilde{f}_* \pi^* j_! M \cong r^i \tilde{f}_! M$$

which induces identity on realization. So it must be an isomorphism in  $\mathcal{M}(Q)$ .  $\square$

**Remark 5.3.2.** *The above construction can be lifted to a functor on derived categories,  $rf_! : D^b \mathcal{M}_c(S; F) \rightarrow D^b \mathcal{M}(Q; F)$ , by defining  $rf_! = (r \bar{f}_*) \circ (j_!)$  using the notation of the proof.*

## 6 Motivic Local Systems

### 6.1 Local Systems

Call an object of  $\Delta(S)$  *tame* if it is of the form  $(\bar{X} - D \rightarrow S, E \cap (\bar{X} - D), i, w)$  such that  $\bar{X} \rightarrow S$  is smooth and projective, and  $D + E$  is a divisor such that any intersection of components is smooth over  $S$  (we will refer to this condition as having relative normal crossings). We usually just write  $E$  instead of  $E \cap (\bar{X} - D)$  above. Such a pair  $(\bar{X} - D \rightarrow S, E)$  is a fibre bundle, so it is controlled.

It is easy to see that  $\Delta_{tame}(S) \subset \Delta_{eq}(S, \{S\})$ .

**Definition 6.1.1.** *The category of premotivic local systems  $\mathcal{PM}_{ls}(S; F) = \text{End}^\vee(H|_{\Delta_{tame}(S)})\text{-comod}$ . The category of motivic local systems  $\mathcal{M}_{ls}(S; F)$  is obtained by forming the associated stack as in section 4.1.*

We note the following properties which are either immediate consequences of what has been said or easily checked.

1.  $\mathcal{M}_{ls}(S) \subset \mathcal{M}_{eq}(S, \{S\})$  is an abelian subcategory.
2. The realizations  $R_B$  and  $R_{et}$  take  $\mathcal{M}_{ls}(S)$  to the categories of locally constant sheaves for the classical and étale topologies, and they factor through  $\mathcal{M}_{ls}(S)$ .
3. The tensor product given earlier restricts to a product  $\mathcal{M}_{ls}(S) \times \mathcal{M}_{ls}(S) \rightarrow \mathcal{M}_{ls}(S)$ . (The key point is that  $\mathcal{M}_{ls}$  is equivalent to comodules over the restriction of  $\text{End}^\vee$  to  $\Delta_{cell} \cap \Delta_{tame}$ .) This induces a product on the stacks  $\mathcal{M}_{ls}(S) \times \mathcal{M}_{ls}(S) \rightarrow \mathcal{M}_{ls}(S)$

By item 2 above, we see that  $\mathcal{M}_{ls}(S)$  is strictly contained in  $\mathcal{M}(S)$  in general. However, we do observe the following:

**Theorem 6.1.2.** *When  $S = \text{Spec} k$ ,  $\mathcal{M}(S; F)$  and  $\mathcal{M}_{ls}(S; F)$  are equivalent.*

*Proof.* By theorem 4.4.2,  $\mathcal{M}(\text{Spec} k; F)$  is equivalent to  $\mathcal{M}_{cell}(S; F)$ . Given  $(X, Y, i, w) \in \Delta_{cell}(S)$ , by resolution of singularities we can find a tame object such that  $(\tilde{X}, E, i, w)$  and a map  $\pi : \tilde{X} \rightarrow X$  which is an isomorphism over  $X - Y$  and such that  $E = \pi^{-1}Y$ . Therefore  $h^i(X, Y)(w) \cong h^i(\tilde{X}, E)(w) \in \mathcal{M}_{ls}$ . Again by resolution of singularities, any morphism in  $\Delta_{cell}$  can be lifted to a morphism in  $\Delta_{tame}$ . This together with lemma 2.2.5 implies the theorem.  $\square$

We outline the construction of Gysin maps, which will be needed later. Given a smooth subscheme  $\bar{Y} \rightarrow S$  of  $\bar{X}$  transverse to  $D + E$  with relative dimension  $m$ . Set  $c = n - m$ . Then the Gysin homomorphism on cohomology

$$H_S^i(\bar{Y} - D, E) \rightarrow H_S^{i+2c}(\bar{X} - D, E)$$

can be defined simply by dualizing the restriction under Poincaré duality. However, this description is not very convenient. A better alternative is to define this via a deformation to the normal bundle as in [BFM]. Let  $\tilde{X}$  be the blow

up of  $\bar{X} \times \mathbb{A}^1$  along  $\bar{Y} \times \{0\}$ . Let  $\tilde{Y}$  be the strict transform of  $\bar{Y} \times \mathbb{A}^1$ . Let  $\tilde{D}, \tilde{E}$  be the preimages of  $D, E$  in  $\tilde{X}$ . The fibre of the natural map  $\pi : \tilde{X} \rightarrow \mathbb{A}^1$  over  $t \neq 0$  is  $X$ . While the fibre  $\pi^{-1}0$  is the union of the projectivized normal bundle  $p : \mathbb{P}(N \oplus \mathcal{O}_{\tilde{Y}}) \rightarrow \tilde{Y}$  and the blow up  $B$  of  $\tilde{X}$  along  $\tilde{Y}$ . Let  $\tau = c_1(\mathcal{O}_{\mathbb{P}(N \oplus \mathcal{O})}(1))^c \in H^{2c}(\mathbb{P}(N \oplus \mathcal{O}), \mathbb{P}(N))$ . The Gysin map can then be realized as the composition of the given maps

$$\begin{aligned}
H_S^i(\bar{Y} - D, E) &\xrightarrow{p^*} H_S^i(p^{-1}Y - p^{-1}D, p^{-1}E) \\
&\xrightarrow{\cup \tau} H_S^{i+2c}(p^{-1}Y - p^{-1}D, p^{-1}E) \\
&\xleftarrow{\cong} H_S^{i+2c}(\pi^{-1}(0) - \tilde{D}, \pi^{-1}(0) \cap \tilde{E} \cup B) \\
&\xleftarrow{\cong} H_S^{i+2c}(\tilde{X} - \tilde{D}, \tilde{E} \cup B) \\
&\xrightarrow{\pi_t^*} H_S^{i+2c}(\bar{X} - D, E)
\end{aligned}$$

The second description yields a motivic Gysin map

$$h_S^i(\bar{Y} - D, E) \rightarrow h_S^{i+2c}(\bar{X} - D, E)(c) \quad (12)$$

We can define the Gysin morphism

$$h_S^i(\bar{Y} - f^{-1}D, f^{-1}E) \rightarrow h_S^{i+2c}(\bar{X} - D, E)(c)$$

for an arbitrary map  $f : \bar{Y} \rightarrow \bar{X}$  as the composition of the Gysin morphism associated to the inclusion of graph of  $\Gamma_f \subset \bar{Y} \times \bar{X}$  followed by a Künneth projection.

When,  $S$  is smooth let  $VMHS(S_{l,an})$  denote the category of rational variations of mixed Hodge structures on  $S_{l,an}$ , which are admissible in the sense of Steenbrink and Zucker [SZ] and Kashiwara [K]. In a nutshell, an object of this category consists of a filtered local system  $(V, W)$  together with a compatible bifiltered vector bundle with connection  $(\mathcal{V} \cong V \otimes_{\mathcal{O}_S} W, F, \nabla)$  subject to the appropriate axioms (Griffith's transversality...). For the precise conditions, see [PS, sect. 14.4.1] or the above references. Given  $(X = \bar{X} - D \rightarrow S, E, i, 0) \in \Delta_{tame}$ , we can construct an admissible variation as follows:

$$\left\{ \begin{array}{l}
V = H_S^i(X, E \cap X; \mathbb{Q}) \\
\mathcal{V} = \mathbb{R}f_* \Omega_{\bar{X}/S}^\bullet(\log D + E)(-E) \\
F^p = \text{im } \mathbb{R}f_* \Omega_{\bar{X}/S}^{\geq p}(\log D + E)(-E) \\
W_q = \text{im } \mathbb{R}f_* W_q \Omega_{\bar{X}/S}^\bullet(\log D + E)(-E) \\
\nabla = \text{Gauss-Manin connection}
\end{array} \right.$$

This is given in [SZ], when  $E = \emptyset$ . The general case is easily reduced to this via the resolution

$$j_{X,E!} \mathbb{Q}_{X-E} \rightarrow \mathbb{Q}_X \rightarrow \bigoplus \mathbb{Q}_{E_i} \rightarrow \bigoplus \mathbb{Q}_{E_i \cap E_j} \dots$$

where  $E = \cup E_i$  is the decomposition into irreducible components. We can extend this to arbitrary objects  $(X = \bar{X} - D \rightarrow S, E, i, w) \in \Delta_{tame}$  by tensoring

the above variation with  $\mathbb{Q}(w)$ . This construction is easily checked to yield a functor  $\Delta_{\text{tame}}(S)^{op} \rightarrow VMHS(S)$ . Thus we get

**Example 6.1.3.** *an exact faithful Hodge realization functor*

$$R_{\iota, H} = R_H : \mathcal{M}_{ls}(S; \mathbb{Q}) \rightarrow VMHS(S_{\iota, an})$$

*This functor is compatible with tensor product. This coincides with the Hodge realization constructed earlier, restricted  $\mathcal{M}_{ls}(S; \mathbb{Q})$ , once we identify  $VMHS(S) \subset \text{Cons-MHM}(S)$ .*

One of the consequences of the admissibility conditions mentioned above is the following removable singularities theorem: An admissible variation extends from a Zariski open to the whole variety if the underlying local system extends. Using this, it is possible to prove a stronger statement that  $R_H$  extends to all of  $\mathcal{M}_{ls}(S)$ .

We can define a system of realizations on  $S$  by following the usual pattern [De2, J1]. Here we outline the construction. A “locally constant” or more correctly lisse  $\ell$ -adic sheaf  $V$  on  $S_{et}$  corresponds to a representation of the algebraic fundamental group  $\pi_1^{et}(S) \rightarrow GL_N(\mathbb{Q}_\ell)$ . Composing this with the canonical map from the topological fundamental group  $\kappa : \pi_1(S_{\iota, an}) \rightarrow \pi_1^{et}(S)$  results in a local system  $\kappa_\iota^* V$  of  $\mathbb{Q}_\ell$ -modules on  $S_{\iota, an}$ . By a system of realizations we will mean

1. A collection of locally constant  $\ell$ -adic sheaves  $V_\ell$  on  $S_{et}$ , for each prime  $\ell$ . Each  $V_\ell$  should be mixed in the sense that they carry weight filtrations.
2. A collection of variations of mixed Hodge structures  $V_\iota$  on  $S_{\iota, an}$  indexed by embeddings of  $\iota : k \hookrightarrow \mathbb{C}$ .
3. Compatibility isomorphisms  $\kappa_\iota^* V_\ell \cong V_\iota \otimes \mathbb{Q}_\ell$  respecting weight filtrations.

These form a  $\mathbb{Q}$ -linear abelian category  $SR(S)$ . An appeal to corollary 2.2.10 and the comparison theorem (appendix B) yields a realization functor  $R_{SR} : \mathcal{M}_{ls}(S, \mathbb{Q}) \rightarrow SR(S)$  which combine all of the previous realizations into one. Thus  $\mathcal{M}_{ls}$  gives a finer theory than motives built from systems of realizations.

## 6.2 Duality

The goal of this section is to prove:

**Theorem 6.2.1.**  *$\mathcal{M}_{ls}(S; F)$  is a neutral Tannakian category over  $F$*

**Corollary 6.2.2.**  *$\mathcal{M}_{ls}(S, \mathbb{Q})$  is equivalent to the category of representations of a proalgebraic group (which we refer to as the Tannakian dual of this category).*

To be more explicit, after choosing a base point  $s \in S(\bar{k})$ , we obtain a so called fibre functor  $F_s : \mathcal{M}_{ls}(S, \mathbb{Q}) \rightarrow \mathbb{Q}\text{-mod}$  given as the composition of  $R_B$  with the stalk at  $s$ . Setting  $\pi_1^{mot}(X, s)$  to the group of tensor automorphisms of  $F_s$ , we have that  $\pi_1^{mot}(X, s)$  is proalgebraic and that  $\mathcal{M}_{ls}(S, \mathbb{Q})$  is equivalent

to the category of representations of it. The methods of [Ar2] show that this carries more structure, but the details will be spelled out elsewhere.

As to the theorem's proof, we know that  $\mathcal{M}_{ls}(S; F)$  is a tensor category over  $F$  with a tensor preserving fibre functor. What remains to be proven is that every object has a dual. By proposition 2.4.3 it is enough to construct duals for objects of the graph  $\Delta_{tame}(S)$ . We will show that

$$h_S^i(\bar{X} - D, E)(w)^\vee = h_S^{2n-i}(\bar{X} - E, D)(-w + n) \quad (\text{Dual})$$

where  $n$  is the relative dimension of  $\bar{X} \rightarrow S$ . As first step, we note the following form of Poincaré duality.

**Lemma 6.2.3.** *There is a pairing*

$$H_S^i(\bar{X} - D, E) \otimes H_S^{2n-i}(\bar{X} - E, D) \rightarrow F_S$$

which is perfect in the sense that it induces an isomorphism of local systems

$$H_S^i(\bar{X} - D, E) \cong H_S^{2n-i}(\bar{X} - E, D)^*$$

*Proof.* This follows from Verdier duality [I]

$$\begin{aligned} H_S^i(\bar{X}, \mathbb{R}j_{(\bar{X}, D)*}j_{(\bar{X}, E)!}F) &\cong H_S^{-i}(\bar{X}, D\mathbb{R}j_{(\bar{X}, D)*}j_{(\bar{X}, E)!}F)^* \\ &\cong H_S^{-i}(j_{(\bar{X}, D)!}\mathbb{R}j_{(\bar{X}, E)*}F[2n])^* \cong H_S^{2n-i}(\bar{X} - E, D)^* \end{aligned}$$

□

The next task is to realize the above pairing geometrically by a morphism of  $\mathcal{M}_{ls}$ . When  $D = E = \emptyset$ , we can take the cup product pairing which is induced by the diagonal embedding into the product. In general, we need to blow up the product to get a well defined diagonal. Set  $Y = \bar{X} \times \bar{X}$ ,  $D_1 = D \times \bar{X}$ ,  $D_2 = \bar{X} \times D$ ,  $E_1 = E \times \bar{X}$  and  $E_2 = \bar{X} \times E$ . Let  $\tilde{Y}$  be obtained by blowing up  $Y$  along  $D_1 \cap D_2$  and then along the intersection of the strict transforms of  $E_1$  and  $E_2$ . Let  $G$  be the exceptional divisor of  $\tilde{Y} \rightarrow Y$ . Denote the strict transforms of  $D_i, E_j$  by  $\tilde{D}_i, \tilde{E}_j$ . The diagonal embedding  $\bar{X} \rightarrow Y$  extends to an embedding of  $d: \bar{X} \rightarrow \tilde{Y}$  (it is not necessary to blow up  $X$  since  $D$  and  $E$  are already divisors). The image of  $d$  is disjoint from  $\tilde{D}_i, \tilde{E}_j$  and  $d^{-1}G \subseteq D \cup E$ . We define

$$\epsilon: h_S^i(\bar{X} - D, E) \otimes h_S^{2n-i}(\bar{X} - E, D)(n) \rightarrow F_S$$

by the composition of

$$\begin{aligned} h_S^i(\bar{X} - D, E) \otimes h_S^{2n-i}(\bar{X} - E, D) &\rightarrow h_S^{2n}(Y - (D_1 \cup E_2), E_1 \cup D_2) \\ &\rightarrow h_S^{2n}(\tilde{Y} - (\tilde{D}_1 \cup \tilde{E}_2 \cup G), \tilde{E}_1 \cup \tilde{D}_2) \\ &\xleftarrow{\cong} h_S^{2n}(\tilde{Y} - (\tilde{D}_1 \cup \tilde{E}_2), \tilde{E}_1 \cup \tilde{D}_2 \cup G) \\ &\rightarrow h_S^{2n}(\bar{X}, E \cup D) \\ &\xrightarrow{\cong} h_S^{2n}(\bar{X}) \cong F(-n) \end{aligned}$$

after twisting by  $F(n)$ . The middle isomorphism is excision. For the last isomorphism, by projection we can reduce to the case  $X = \mathbb{P}_S^n$  and then to  $X = (\mathbb{P}_S^1)^n$ , where it follows from Künneth.

To construct  $\delta$ , we dualize the above description using Gysin maps in place of pull backs:

$$\begin{aligned}
F_S = h_S^0(\bar{X}, E \cup D) &\rightarrow h_S^{2n}(\tilde{Y} - (\tilde{D}_1 \cup \tilde{E}_2), \tilde{E}_1 \cup \tilde{D}_2 \cup G)(n) \\
&\xrightarrow{\cong} h_S^{2n}(\tilde{Y} - (\tilde{D}_1 \cup \tilde{E}_2 \cup G), \tilde{E}_1 \cup \tilde{D}_2)(n) \\
&\rightarrow h_S^{2n}(Y - (D_1 \cup E_2), E_1 \cup D_2)(n) \\
&\rightarrow h_S^i(\bar{X} - D, E) \otimes h_S^{2n-i}(\bar{X} - E, D)(n)
\end{aligned}$$

To prove (Dual), we have to establish equations (D1) and (D2). It is enough to verify these on the corresponding vector spaces  $H_S^i(\bar{X} - D, E)_s, H_S^{2n-i}(\bar{X} - E, D)_s$ , and this becomes an exercise in linear algebra. If  $e_j$  is a basis of the first space, and  $e^j$  the dual basis of the second, then

$$\begin{aligned}
\delta(1) &= \sum_{\ell} e^{\ell} \otimes e_{\ell} \\
\epsilon(\sum a_{j\ell} e_j \otimes e^{\ell}) &= \sum a_{jj}
\end{aligned}$$

Therefore

$$(\epsilon \otimes id) \circ (id \otimes \delta)(\sum a_j e_j) = (\epsilon \otimes id)(\sum a_j e_j \otimes e^{\ell} \otimes e_{\ell}) = \sum a_{\ell\ell} e_{\ell}$$

proves (D1). The remaining equation is similar.

### 6.3 Pure Objects and Weights

We work in  $\mathcal{M}(S, \mathbb{Q})$  throughout this section. Let  $f : X \rightarrow S$  be a smooth projective map of relative dimension  $n$ . Fix an embedding  $X \subset \mathbb{P}_S^N$ . The standard generator  $c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^N)$  induces an isomorphism  $\mathbb{Q}_S(0) \cong h_S^2(\mathbb{P}_S^N)(1)$ . This yields a map  $\mathbb{Q}_S(0) \rightarrow h_S^2(X)(1)$  by restriction. Cupping with this induces the Lefschetz operator  $\ell : h_S^i(X) \rightarrow h_S^{i+2}(X)(1)$ . The isomorphism

$$\ell^i : h_S^{n-i}(X) \xrightarrow{\sim} h_S^{n+i}(X)(i)$$

follows from the usual hard Lefschetz theorem on the corresponding sheaves. Therefore we get, as usual, the Lefschetz decomposition  $h_S^i(X) = \oplus \ell^k p^{i-2k}(X)(-k)$ , where  $p^i(X) = h_S^i(X) \cap \ker \ell^{n-i+1}$ . This allows us to define the Hodge involution  $* = *_H$  on  $h_S^*(X) = \oplus h_S^i(X)$  by the formula in [A1, pp 10-11]. Note that the induced involution on the cohomology of a fiber  $H^*(X_s, \mathbb{C})$  coincides with the Hodge star operator with respect to the Fubini-Study metric (up to a factor and complex conjugation) [loc. cit.]

**Proposition 6.3.1.** *The algebra  $End(h_S^*(X))$  is semisimple.*



*Proof.* Set  $a' = *a^t*$ , where  $a^t$  is the transpose (c.f. [K, 1.3]). With the help of the Hodge index theorem, we see that the bilinear form  $\text{trace}(ab')$  is positive definite (compare [K, p. 381]). Then the criterion of [K, 3.13] shows that the algebra is semisimple.  $\square$

**Definition 6.3.2.** *Call an object of  $\mathcal{M}_{l_S}(S)$  pure (of weight  $i$ ) if it is a finite sum of summands of motives  $h_S^*(X)$  (or  $h_S^i(X)$ ) with  $X \rightarrow S$  smooth and projective. Let  $\mathcal{M}_{\text{pure}}(S) \subset \mathcal{M}_{l_S}(S)$  ( $\mathcal{M}_{\text{pure},i}(S) \subset \mathcal{M}_{l_S}(S)$ ) be the full subcategory of pure objects.*

**Theorem 6.3.3.**  *$\mathcal{M}_{\text{pure}}(S, \mathbb{Q})$  and  $\mathcal{M}_{\text{pure},i}(S, \mathbb{Q})$  are semisimple abelian subcategories of  $\mathcal{M}_{l_S}(S, \mathbb{Q})$ . The Hodge realization  $R_H$  takes  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q})$  (respectively  $\mathcal{M}_{\text{pure},i}(S, \mathbb{Q})$ ) to the category of pure polarizable variations of Hodge structure  $HS(S)$  (of weight  $i$  respectively) There is a direct sum decomposition  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q}) = \bigoplus_i \mathcal{M}_{\text{pure},i}(S, \mathbb{Q})$ , i.e. every object and morphism on the left decomposes into a sum as indicated. Furthermore  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q})$  is a Tannakian subcategory.*

*Proof.* These are abelian and semisimple by [J2, lemma 2] and the previous proposition. The second statement is clear. The third statement follows immediately from the previous two. It is easy to see from the constructions that  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q})$  is closed under tensor product and duals.  $\square$

**Corollary 6.3.4.** *The Tannakian dual of  $\mathcal{M}_{\text{pure}}(S, \mathbb{Q})$  is proreductive.*

**Theorem 6.3.5.** *There are exact functors  $gr_j : \mathcal{M}_{l_S}(S, \mathbb{Q}) \rightarrow \mathcal{M}_{\text{pure},j}(S, \mathbb{Q})$  which splits the inclusions  $\mathcal{M}_{\text{pure},j}(S, \mathbb{Q}) \subset \mathcal{M}_{l_S}(S, \mathbb{Q})$ . These are compatible with the Hodge realizations in the sense that  $R_H gr_j = Gr_j^W R_H$ .*

*Proof.* It is enough to define  $gr = \bigoplus_j gr_j$ , and then set  $gr_j$  to the composition of this with the projection  $\mathcal{M}_{\text{pure}}(S) \rightarrow \mathcal{M}_{\text{pure},j}(S)$ .

Fix a smooth projective map  $\bar{X} \rightarrow S$  with a relative normal crossing divisor  $D + E$ . Then  $H_S^*(\bar{X} - D, E, \mathbb{C})$  can be computed by taking the direct image of the double complex

$$\Omega_{\bar{X}/S}^\bullet(\log D) \rightarrow \bigoplus_i \Omega_{E_i/S}^\bullet(\log D) \rightarrow \bigoplus_{i < j} \Omega_{E_i \cap E_j/S}^\bullet(\log D) \rightarrow \dots$$

When restricted to the fibres, this complex forms part of a differential graded cohomological mixed complex [De1]. From this we can deduce a spectral sequence associated to the diagonal filtration (c.f. [De1, 7.1.6, 8.1.19.1])

$$E_1^{-a,b} = \bigoplus_{p+2r=b, q-r=-a} H_S^p(Y_q^{(r)}) \Rightarrow H_S^{b-a}(\bar{X} - D, E, \mathbb{Q}) \quad (13)$$

where

$$Y_q^{(r)} = \begin{cases} D^{(r)} & \text{if } q = 0 \\ E^{(q-1)} \cap D^{(r)} & \text{if } q > 0 \end{cases}$$

and  $D^{(r)}$  and  $E^{(r)}$  are disjoint unions of  $r + 1$ -fold intersections of components of  $D$  and  $E$ . We see also that (13) degenerates at  $E_2$ , and the induced filtration on the abutment is the weight filtration. For later use, we record the precise formula for  $W_{i+k}H_S^i(\bar{X} - D, E, \mathbb{C})$

$$\mathrm{im} H^i(W_k \Omega_{\bar{X}/S}^\bullet(\log D) \rightarrow \bigoplus W_{k+1} \Omega_{E_i/S}^\bullet(\log D) \rightarrow \dots) \quad (14)$$

The differentials of (13) are sums of restrictions and Gysin maps. So we can regard this as a spectral sequence of variations of Hodge structures

$$E_1^{-a,b} = \bigoplus_{p+2r=b, q-r=-a} H_S^p(Y_q^{(r)})(-r) \Rightarrow Gr^W H_S^{b-a}(X, \mathbb{Q}) \quad (15)$$

As noted above, the differentials are sums of restrictions and Gysin maps. The motivic versions of Gysin maps were defined in (12) of section 6.1. Thus we can form a graded complex of motives in  $\mathcal{M}_{pure}$

$$e^{-a,b} = \bigoplus_{p+2r=b, q-r=-a} h_S^p(Y_q^{(r)})(-r)$$

which maps to the left side of (15) under  $R_H$ .

We would like to take  $gr(h^i(\bar{X} - D, E))$  to be the sum  $\bigoplus h^{-a}(e^{\bullet, i+\bullet})$ , but at the moment this not well defined. It depends on the choice of compactification, so we denote it by  $G^i(\bar{X}, D, E)$ . We build a graph  $\tilde{\Delta}(S)$ , with a forgetful functor  $\pi : \tilde{\Delta}(S) \rightarrow \Delta(S)$ , whose objects consist of such compactifications, along with labels  $i, w \in \mathbb{Z}$ . Any two compactifications are dominated by a third. Therefore the fibres of  $\pi$  are connected. From corollary 2.2.4, it follows that  $End^\vee(H \circ \pi) \cong End^\vee(H)$ . Let  $C$  be the category of triples  $(A, B, \phi)$ , where  $A \in \mathcal{M}_{ls}(S)$ ,  $B \in \mathcal{M}_{pure}(S)$ , and  $\phi : Gr_*^W R_H A \cong R_H B$ . There is an exact faithful functor  $U_2 : C \rightarrow \mathbb{Q}^2\text{-mod}$  taking  $(A, B, \phi)$  to  $U(A) \times U(B)$ , where  $U : \mathcal{M}_{ls}(S) \rightarrow \mathbb{Q}\text{-mod}$  is the forgetful functor. We have a functor from  $H^\# : \tilde{\Delta}(S) \rightarrow C$  sending a labelled compactification  $(\bar{X}, i, w)$  to  $(h^i(\bar{X} - D, E)(w), G^i(\bar{X}, D, E)(w), \phi)$ , where  $\phi$  is the natural isomorphism of Hodge realizations from (15). We can form the category  $\mathcal{PM}^\#$  of comodules over  $End^\vee(U_2 \circ H^\#)$ . This has a natural projection  $p : \mathcal{PM}^\# \rightarrow End^\vee(H \circ \pi)\text{-comod}$ . There is an equivalence  $\mathcal{PM}^\# / \ker p \sim \mathcal{PM}(S)$ . We have a functor  $G : \mathcal{PM}^\# \rightarrow \mathcal{M}_{pure}(S)$  which factors through this equivalence, and this yields  $gr$ .  $\square$

This leads to a theory of weights in  $\mathcal{M}_{ls}$ . Let us say that an object  $M \in \mathcal{M}_{ls}(S, \mathbb{Q})$  has weight(s) in  $I \subset \mathbb{Z}$  if  $gr_j M = 0$  for  $j \notin I$ . For  $M \in \mathcal{M}_{ls}(S, \mathbb{Q})$ . Define  $W_k M$  to be the maximal subobject of  $M$  with weights  $\leq k$ . Note that this exists because  $\mathcal{M}_{ls}(S, \mathbb{Q})$  embeds into  $\mathbb{Q}\text{-mod}$ , so it is noetherian.

**Theorem 6.3.6.** *For all  $k$ , one has*

1.  $W_k M / W_{k-1} M \cong gr_k M$ ,
2.  $W_k$  is strictly preserved by morphisms,

3.  $R_H(W_k M) = W_k R_H(M)$ .

We first need:

**Proposition 6.3.7.** *Given  $M \in \mathcal{M}(S)$  and  $j \in \mathbb{Z}$ , there exists  $N \subseteq M$  and  $N' \subseteq M$  so that  $N'$  has weights  $< j$  and  $N/N' \cong gr_j M$ .*

*Proof.* By lemma 2.2.5, we can assume that  $M = h_S^i(\bar{X} - D, E)$  with  $\bar{X}$  smooth, and  $D + E$  a divisor with relative normal crossings. In principle, the proof amounts to realizing the formula for  $W$  given in (14) by a motive. When  $E = 0$ , this is easy to do directly. Let  $D_{(r)}$  denote union of  $(\dim(D/S) - r + 1)$ -fold intersections of components of  $D$ , with  $D_{(-1)} = \emptyset$ . The point is that  $\dim(D_{(r)}/S) = r$ . Then (14) reduces to

$$W_{i+k} H^i(\bar{X} - D, \mathbb{C}) = \text{im } H^i(W_k \Omega_X^\bullet(\log D)) = \text{im } H^i(X - D_{(k)}, \mathbb{C})$$

Therefore  $N$  (respectively  $N'$ ) may be taken as

$$\text{im}[h^i(\bar{X} - D_{k-1}) \rightarrow h^i(\bar{X} - D)]$$

with  $k = j - i$  (respectively  $k = j - i - 1$ ).

The general case, while feasible, is rather messy to write explicitly. So instead, we finish the proof by induction on  $d = \dim(\bar{X}/S)$ . Once we have established this for a given  $d$ , it follows the proposition holds for all motives generated as an abelian category by varieties of dimension at most  $d$ . So now consider the sequence

$$h^{i-1}(E - D) \rightarrow h^i(\bar{X} - D, E - D) \rightarrow h^i(\bar{X} - D)$$

By induction, we can find  $N_E, N'_E \subset h^{i-1}(E - D)$  satisfying the proposition for this motive. Then

$$N = \text{im } N_E + \text{im}[h^i(\bar{X} - D_{(j-i-1)}, E) \rightarrow h^i(\bar{X} - D, E)]$$

$$N' = \text{im } N'_E + \text{im}[h^i(\bar{X} - D_{(j-i-2)}, E) \rightarrow h^i(\bar{X} - D, E)]$$

will satisfy the proposition for  $M$ . □

*Proof of theorem.* Let  $k$  be the least weight of  $M$ . The canonical map  $\iota : gr_k(W_k M) \rightarrow gr_k M$  is a monomorphism, since  $gr_k$  is exact. The previous proposition shows that  $\iota$  is also an epimorphism and hence an isomorphism. Applying the same argument to the quotients  $M/W_j M$  establishes part 1. The filtration  $W_k$  is functorial by construction. Strictness follows by what has just been proved (cf [De1, §1]). Finally part 3 follows immediately from theorem 6.3.5. □

From the construction, we can deduce the following:

**Proposition 6.3.8.** *The total functor  $gr : \mathcal{M}_{ls}(S) \rightarrow \mathcal{M}_{pure}(S)$  is an exact tensor functor.*

**Corollary 6.3.9.** *The Tannkian dual of  $\mathcal{M}_{ls}(S)$  is a semidirect product of the Tannakian dual of  $\mathcal{M}_{pure}(S)$  with another group.*

## 6.4 André's category of motives

André [A1] has given an entirely different construction of pure motives over a field  $k$  that we recall. Given a smooth projective variety  $X \in \text{Var}_k$ , a class in  $H^{2n}(X, \mathbb{Q})$  is called *motivated* cycle of degree  $n$  if it can be expressed as  $p_*(\alpha \cup \beta)$ , where  $\alpha, \beta$  are algebraic cycles on a product  $X \times Y$ , with  $Y$  smooth and projective, and  $p : X \times Y \rightarrow X$  is the projection. Let  $A_{mot}^n(X)$  denote the set of these classes. It contains the space of algebraic cycles and would coincide with it assuming Grothendieck's standard conjectures. André's category of motives  $M_A(k)$  is built by taking as objects triples  $(X, n, p)$  with  $X$  smooth projective,  $n \in \mathbb{Z}$ , and  $p$  an idempotent in the ring of motivated cycles on  $X \times X$ . Morphisms are given by

$$\text{Hom}_{M_A}((X, n, p), (Y, m, q)) = pA_{mot}^{n-m}(X \times Y)q$$

André proved that this category is semisimple Tannakian. The construction of  $M_A$  and this result was extended to more general smooth bases in [AD]. For simplicity, we concentrate on the case of  $S = \text{Spec} k$ . Given a smooth projective variety  $X$ , let  $h_A(X) \in X$  denote the object represented by the triple  $(X, 0, id)$  and  $h_A^i(X)$  by  $(X, n, \pi_i)$ , where  $\pi_i : H^{2 \dim X - i}(X) \otimes H^i(X)$  is Künneth component of the diagonal. Then  $h_A(X) = \oplus h_A^i(X)$ . By construction, we have an exact faithful embedding  $R_B : M_A(k) \rightarrow \mathbb{Q}\text{-mod}$ , sending  $(X, n, p)$  to  $pH^*(X)$ . In particular,  $R_B(h^i(X)) = H^i(X)$ . The category  $M_A$  can be also be constructed by first forming the category *MotCor* of smooth projective varieties and motivated correspondences, and then taking the pseudo-abelian (or idempotent) completion, and then formally inverting Tate.

**Theorem 6.4.1.** *For any smooth  $S$ , the categories  $M_A(S)$  and  $M_{pure}(S, \mathbb{Q})$  are equivalent.*

We will give the proof when  $S = \text{Spec} k$  for simplicity, even though the arguments works in general. We will refer to a cohomology class in  $H^i(X, \mathbb{Q})$  as a *Nori cycle* of weight  $2j$  if it lies in the image of  $\text{Hom}_{M(k)}(\mathbb{Q}(-j), h^i(X))$ .

*Proof.* This is broken into a series of steps.

1. Motivated cycles are Nori cycles.

*Proof.* Algebraic cycles are certainly Nori cycles. In general, motivated cycles are built from algebraic cycles by applying  $*$ ,  $\cup$ ,  $p_*$ . Each of these operations preserves the space of Nori cycles.  $\square$

2. There is an functor  $\iota : M_A \rightarrow M_{pure}$  taking  $h_A^i(X)$  to  $h^i(X)$ . This functor commutes with  $R_B$ .

*Proof.* By 1, the map on objects  $X \rightarrow \oplus h^i(X)$  gives a functor  $\iota' : \text{MotCor} \rightarrow M_{pure}$ . Since  $M(k)$  is abelian and the Tate motive is invertible,  $\iota'$  extends uniquely to a functor  $\iota : M_A \rightarrow M_{pure}$ . The final statement follow more or less automatically from  $R_B(h_A^i(X)) = H^i(X) = R_B(h^i(X))$ .  $\square$

3. There is a functor  $gr_A : M \rightarrow M_A$  taking  $h^i(X) \rightarrow h_A^i(X)$ . This satisfies  $gr_A \circ \iota = id$  and it commutes with  $R_B$ .

*Proof.* The functor  $gr_A$  is constructed by substituting  $M_A$  for  $M_{pure}$  in the proof of theorem 6.3.5.  $\square$

4. The functor  $\iota$  takes simple objects to simple objects.

*Proof.* Suppose that  $M \in M_A$  is simple, but  $\iota(M)$  is not. We may write  $\iota(M) = N_1 \oplus N_2$  with  $N_i \neq 0$ . Then  $M = gr_A(N_1) \oplus gr_A(N_2)$  which leads to a contradiction.  $\square$

5. The set of simple objects of  $M_A$  and  $M_{pure}$  are in one to one correspondence via  $\iota$  and  $gr_A$ .

*Proof.* Write  $h_A^i(X) = \oplus M_j$ , with  $M_j$  simple. Then  $h^i(X) = \oplus \iota(M_j)$  gives a decomposition into simple objects by 4. Since every simple object of  $M_{pure}$  is a summand of some  $h^i(X)$ , with  $X$  smooth and projective, this proves the claim.  $\square$

6. If  $M \in M_A$  is simple, then  $End(M) \cong End(\iota(M))$ .

*Proof.* The map  $g : End(\iota(M)) \rightarrow End(M)$ , induced by  $gr_A$ , is a surjective homomorphism because  $\iota$  gives a splitting. Since  $End(\iota(M))$  is a division ring,  $g$  is necessarily an injection as well.  $\square$

7. Given  $M, N \in M_A$ ,  $Hom(M, N) \cong Hom(\iota(M), \iota(N))$ .

*Proof.* Decompose  $M = \bigoplus M_j^{m_j} \oplus M'$  and  $N = \bigoplus M_j^{n_j} \oplus N'$ , such that  $M_j$  are distinct simple objects and  $M', N'$  have no simple factors in common. Let  $D_j = End(M_j)$ . Then

$$Hom(M, N) = \prod Mat_{n_j m_j}(D_j) = End(\iota(M))$$

$\square$

$\square$

## 7 Nori's Hodge conjecture

### 7.1 Conjecture over $\mathbb{C}$

As is well known, the usual form of the Hodge conjecture is equivalent to the statement that the Hodge realization of homological pure motives is a full faithful embedding [Sv, p 405]. In a nutshell, this comes down to the observation that given smooth projective varieties  $X$  and  $Y$ , a morphism  $\text{Hom}_{MHS}(H^i(X), H^i(Y))$  is a Hodge cycle on  $X \times Y$  and therefore a correspondence, assuming the conjecture. The analogous statement in the present setting is due to Nori.

**Conjecture 7.1.1** (Nori). *The Hodge realization  $R_H : \mathcal{M}(\mathbb{C}, \mathbb{Q}) \rightarrow MHS$  is a full faithful embedding.*

This would imply that the canonical mixed Hodge structure is ‘‘Galois invariant’’ in the following sense: if  $H^i(X) \cong H^i(Y)$  in MHS, then  $H^i(X^\sigma) \cong H^i(Y^\sigma)$  for any  $\sigma \in \text{Aut}(\mathbb{C})$ . Since  $\mathcal{M}(\mathbb{C}) = \mathcal{M}_{ls}(\mathbb{C})$  and  $MHS$  are Tannakian, we can rewrite  $\text{Hom}(A, B) = \text{Hom}(\mathbb{Q}, A^* \otimes B)$ , and reformulate the last conjecture as

**Conjecture 7.1.2.** *The map*

$$\text{Hom}_{\mathcal{M}(\mathbb{C})}(\mathbb{Q}, M) \rightarrow \text{Hom}_{MHS}(\mathbb{Q}, R_H(M))$$

*is surjective for each  $M \in \mathcal{M}(\mathbb{C})$ . In particular, a Hodge cycle on any complex algebraic variety  $X$  is a Nori cycle.*

When  $M$  lies in  $\mathcal{M}_{pure}(\mathbb{C})$ , this is implied by the usual Hodge conjecture, but in general, it is neither weaker nor stronger than the Hodge conjecture. It should be viewed as refinement of Deligne’s conjecture that Hodge cycles are absolute [DMOS]. To understand this from a different perspective, let us recall that the original form of Beilinson’s Hodge conjecture [B1] would imply that the regulator map on the higher Chow group

$$\text{reg} : CH^a(X, b) \otimes \mathbb{Q} \rightarrow \text{Hom}_{MHS}(\mathbb{Q}(-a), H^{2a-b}(X))$$

is surjective for all  $a, b$ . The conjecture is known to be overly optimistic in general (cf [J2]), but it is expected for instance when  $X$  is defined over  $\bar{\mathbb{Q}}$ . The map can be made explicit as follows. An element  $\alpha$  on the left is given a cycle in Bloch’s complex [BI], and so it possesses a fundamental class in

$$\text{reg}(\alpha) \in H^{2a}(X \times \mathbb{A}^b, X \times \partial\mathbb{A}^b)(a) \cong H^{2a-b}(X)(a)$$

where  $\mathbb{A}^b$  is thought of as a simplex with boundary  $\partial\mathbb{A}^b$ . It is clear from this, that we may factor  $\text{reg}$  through

$$\text{Hom}_{\mathcal{M}(\mathbb{C})}(\mathbb{Q}(-a), H^{2a-b}(X)) \rightarrow \text{Hom}_{MHS}(\mathbb{Q}(-a), H^{2a-b}(X))$$

Thus the truth of Beilinson’s conjecture, in cases where it is expected, would imply the truth of Nori’s.

**Theorem 7.1.3** (André [A1]). *Hodge cycles on abelian varieties are motivated.*

So we deduce:

**Corollary 7.1.4.** *Conjecture 7.1.2 holds for any variety whose motive lies in the tensor category generated by abelian varieties.*

## 7.2 Conjecture over general bases

We can formulate an ostensibly stronger form of Nori's conjecture.

**Conjecture 7.2.1.** *Given a smooth complex variety  $S$ ,*

- (a) *the Hodge realization  $R_H : \mathcal{M}_{l_s}(S, \mathbb{Q}) \rightarrow VMHM(S)$  is a full faithful embedding.*
- (b) *if  $M \in \mathcal{M}_{l_s}(S)$ , the map*

$$Hom_{\mathcal{M}_{l_s}(S)}(\mathbb{Q}_S, M) \rightarrow Hom_{VMHS}(\mathbb{Q}_S, R_H(M))$$

*is surjective.*

As above, we note that (a) and (b) are equivalent because  $\mathcal{M}_{l_s}(S)$  is Tannakian. We will prove that the conjectures for a general base follows from the earlier ones. As a first step, let us suppose that we have motive  $M \in \mathcal{M}_{l_s}(S)$ , then by theorem 5.2.1, we have a good direct image  $p_*M = r^0p_*M \in \mathcal{M}(\mathbb{C})$ , where  $p : S \rightarrow Spec \mathbb{C}$  is the canonical map. Restricting the adjunction map  $p^*p_*M \rightarrow M$  to  $s \in S$ , yields map  $p_*M \rightarrow M_s$ . Under Betti realization  $R_B(p_*M) \rightarrow R_B(M_s)$  can be identified with the inclusion of the subspace of  $\pi_1(S, s)$ -invariants of the fibre of the local system  $R_B(M)$ . As an aside, we observe that this leads to a direct construction for basic examples:

**Lemma 7.2.2.** *If  $M = h_S^i(X, E)$ , then*

$$p_*M \cong \text{im}[h^i(X, E) \rightarrow h^i(X_s, E_s)]$$

*Proof.* Let  $I$  denote the right hand side. Consider the inclusion  $X \rightarrow X \times S$  of  $S$ -schemes given by the graph of  $X \rightarrow S$ . This induces a map  $h^i(X, E) = p_*h_S^i(X \times S, E \times S) \rightarrow p_*M$ , which is a morphism  $I \rightarrow p_*M$ . It suffices to prove that this is an isomorphism of Betti realizations. For this, apply the global invariant cycle theorem [BBD, 6.2.8]:

$$R_B(I) = \text{im}[H^i(X, E) \rightarrow H^i(X_s, E_s)] = H^i(X_s, E_s)^{\pi_1(S, s)}$$

□

**Theorem 7.2.3.** *Suppose that  $S$  is smooth and connected with a point  $s \in S$ . Given  $M \in \mathcal{M}_{l_s}(S)$ , if conjecture 7.1.2 holds for  $p_*M$  then conjecture 7.2.1(b) holds for  $M$ .*

*Proof.* Let us suppose that conjecture 7.1.2 holds for  $I = p_*M$ . Any morphism  $\gamma \in \text{Hom}(\mathbb{Q}_S, R_H(M))$  gives a  $\pi_1(S, s)$  invariant weight Hodge cycle on  $H^i(X_s, E_s)$ . Thus  $\gamma$  lies in  $\text{Hom}(\mathbb{Q}_S, R_H(I))$ . So it must come from a morphism  $\mathbb{Q} \rightarrow I$  by our assumption. This induces a map of pullbacks  $\mathbb{Q}_S \rightarrow p^*I$ , which when composed with the adjunction map  $\epsilon : p^*I \rightarrow M$  gives the desired lift of  $\gamma$  to  $\gamma' \in \text{Hom}_{\mathcal{M}_{1s}(S)}(\mathbb{Q}_S, M)$ .  $\square$

**Corollary 7.2.4.** *Conjecture 7.1.2 implies conjecture 7.2.1.*

**Corollary 7.2.5.** *Conjecture 7.2.1(b) for any motive that lies in the tensor category generated by relative smooth curves over  $S$ .*

By a similar argument we obtain an analogue of Deligne’s “principle B” [DMOS] in the theory of absolute Hodge cycles.

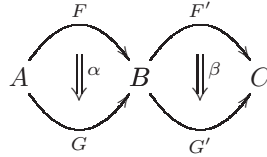
**Proposition 7.2.6.** *Given a tame family  $(f : X \rightarrow S, E)$ , a  $\pi_1(S, s)$ -invariant Nori cycle in  $H^i(X_s, E_s)$  yields, under parallel transport, a Nori cycle in every fibre  $H^i(X_t, E_t)$ .*

*Proof.* A  $\pi_1(S, s)$ -invariant Nori cycle on  $H^i(X_s, E_s)$  induces a morphism  $\mathbb{Q}(j) \rightarrow p_*h^i(X, E)$ . This can be specialized to any fibre.  $\square$

## Appendices

### A 2-categories

We will generally take the view that a category *is* the same as any other category equivalent to it. This needs some elaboration. The category  $Cat$  of all small categories is a 2-category [M]. Among other things, this means that the set of functors  $\text{Hom}_{Cat}(C, D)$  is itself a category, where the morphisms are natural transformations. In this setting, functors are usually called 1-morphisms and natural transformations 2-morphisms. Each kind of morphism can be composed as usual. This is denoted by  $\circ$ . There are identities denoted by  $1_X$  etc. There is another kind of composition for adjacent 2-morphisms, denoted here by  $\diamond$ . Given objects  $A, B, C$ , 1-morphisms  $F, F', G, G'$  and 2-morphisms  $\alpha, \beta$  as indicated below





The composition  $\beta \diamond \alpha$  sits as follows

$$\begin{array}{ccc}
 & F' \circ F & \\
 & \curvearrowright & \\
 A & \Downarrow \beta \diamond \alpha & C \\
 & \curvearrowleft & \\
 & G' \circ G & 
 \end{array}$$

It is simply given as the composition

$$F'(F(x)) \xrightarrow{F'(\alpha_x)} F'(G(x)) \xrightarrow{\beta_{G(x)}} G'(G(x))$$

This operation is associative, and the additional identities

$$\begin{aligned}
 \alpha \diamond 1_{1_A} &= 1_{1_A} \diamond \alpha = \alpha, & 1_F \diamond 1_G &= 1_{F \circ G} \\
 (\beta' \circ \beta) \diamond (\alpha' \circ \alpha) &= (\beta' \diamond \alpha') \circ (\beta \diamond \alpha)
 \end{aligned}$$

are satisfied.

In  $Cat$ , we can either require that equations (or diagrams) hold (or commute) strictly, i.e. on the nose, or only up to a natural isomorphism. The latter is frequently the more usual occurrence. Recall, for example, that categories  $C$  and  $D$  are equivalent (respectively isomorphic) if there are functors  $F : C \rightarrow D$  and  $G : D \rightarrow C$  such that  $F \circ G \cong 1_D$  and  $G \circ F \cong 1_C$  (respectively  $F \circ G = 1_D$  and  $G \circ F = 1_C$ ). Given a category  $C$ , a pseudofunctor  $F : C \rightarrow Cat$  is an assignment of objects to objects, and morphisms to 1-morphisms, together with natural isomorphisms,  $\epsilon_c : F(1_c) \cong 1_{F(c)}$  and  $\eta_{f,g} : F(f) \circ F(g) \cong F(f \circ g)$ . These are required to satisfy certain commutivities that ensure that any two isomorphisms

$$F(f_1) \circ F(f_2) \circ \dots \circ F(f_n) \cong F(f_1 \circ f_2 \circ \dots \circ f_n)$$

built from  $\eta, \epsilon$  coincide. It suffices to check this for  $n \leq 3$ . Contravariant pseudofunctors are simply pseudofunctors on  $C^{op}$ .

There are number of related notions of colimit (= direct limit) of categories. We single out the notion that is most useful for us and refer to it as a 2-colimit, although “pseudo-colimit” or something like that may conform better to current usage. To simplify matters, we discuss 2-colimits in the filtered setting where things are easier (cf. [SGA4, exp VI §6]); this reference also gives a more general construction. A category  $D$  is filtered if for any two objects  $d_1, d_2$ , there exists an object  $d_3$  and morphisms  $d_1 \rightarrow d_3, d_2 \rightarrow d_3$ , and for any two parallel morphisms  $f, g : d_1 \rightarrow d_2$ , there exists a morphism  $h : d_2 \rightarrow d_3$  such that  $hf = hg$ . For example, a partially ordered set is filtered precisely when it is directed. Given a pseudofunctor  $F : D \rightarrow Cat$  with  $D$  filtered, the 2-colimit  $L = 2\text{-}\varinjlim_d F(d)$  is the category whose set of objects is the disjoint union

$$ObL = \coprod_d ObF(d)$$

The set of morphisms from  $A \in \text{Ob}F(d_1)$  to  $B \in \text{Ob}F(d_2)$  is given by the filtered colimit

$$\varinjlim_{f:d_1 \rightarrow d_3, g:d_2 \rightarrow d_3} \text{Hom}_{F(d_3)}(F(f)(A), F(g)(B))$$

We can see that there is a family of 1-morphisms  $\{F(d) \rightarrow L\}$  such that for  $f \in \text{Hom}(d, d')$

$$\begin{array}{ccc} F(d) & \longrightarrow & L \\ F(f) \downarrow & \nearrow & \\ F(d') & & \end{array}$$

commutes up to natural isomorphism. Moreover  $L$  would be universal in the sense that for any category  $L'$  with a family  $\{F(d) \rightarrow L'\}$  as above, there is a unique 1-morphism  $L \rightarrow L'$  such that the appropriate diagrams nonstrictly commute. We really want to consider  $2\text{-}\varinjlim_d F(d)$  only up to equivalence of categories. In practice, there may be other representations of the colimit which are more natural than the original construction.

**Lemma A.0.7.** *Suppose that  $C_i \subset C$  is a directed family of subcategories of a given category. Then  $2\text{-}\varinjlim C_i$  is equivalent to the directed union  $\bigcup C_i$  which the category having  $\bigcup \text{Ob}C_i$  and  $\bigcup \text{Mor}C_i$  as its set of objects and morphisms.*

**Example A.0.8.** *Suppose  $E$  is a coalgebra over a field  $F$ . We can express it as directed union, and therefore a 2-colimit, of finite dimensional coalgebras  $E = \varinjlim E_i$ .*

From the earlier description, it is not difficult to deduce the following:

**Proposition A.0.9.** *Let  $F$  be a pseudofunctor from a filtered category  $D$  to the 2-category of abelian categories. Suppose that  $F(f)$  is exact for each  $f \in \text{Mor}D$ . Then  $2\text{-}\varinjlim_d F(d)$  is abelian and the functors  $F(d') \rightarrow 2\text{-}\varinjlim_d F(d)$  are exact.*

*Sketch.* It is clear that  $L = 2\text{-}\varinjlim_d F(d)$  and the functors  $F(d') \rightarrow L$  are additive. Given a morphism in  $L$  represented by  $f : A \rightarrow B$  in  $F(d)$ ,  $\ker(f)$ ,  $\text{coker}(f)$  and  $A \twoheadrightarrow \text{im}(f) \twoheadrightarrow B$  represents the kernel, cokernel and image factorization in  $L$ .  $\square$

In a similar vein:

**Proposition A.0.10.** *Let  $F$  be a pseudofunctor from a filtered category  $D$  to the 2-category of triangulated categories (with  $t$ -structure) and (exact) triangulated functors. Then  $2\text{-}\varinjlim_d F(d)$  carries the structure of a triangulated category (with  $t$ -structure) so that the functors  $F(d') \rightarrow 2\text{-}\varinjlim_d F(d)$  are triangulated (and exact).*

## B Comparison theorem

Let  $X$  be a  $\mathbb{C}$ -variety. Define the site  $X_{cl}$  with objects given by local homeomorphisms  $U \rightarrow X_{an}$  and coverings are surjective families  $\{U_i \rightarrow U\}$ . Then there is an obvious map of sites  $X_{cl} \rightarrow X_{an}$ , which induces an equivalence of the categories of sheaves [SGA4, exp XI §4]. In particular, the cohomologies are the same. There is a canonical morphism of sites  $\epsilon : X_{cl} \rightarrow X_{et}$  which induces a map from étale to classical cohomology.

Since étale cohomology does not work properly for nontorsion coefficients, we start with finite coefficients, and then take the limit. Choose  $N > 0$ . A sheaf of  $\mathbb{Z}/N\mathbb{Z}$ -modules is constructible for either topology if there is a decomposition of  $X$  into Zariski locally closed sets, for which the restrictions are locally constant. The pullback  $\epsilon^*$  preserves constructibility. The following comparison theorem is given in [SGA4, exp XVI, thm 4.1; exp XVII, thm 5.3.3]:

**Theorem B.0.11.** *Suppose that  $f : X \rightarrow Y$  is a morphism of  $\mathbb{C}$  varieties, and that  $\mathcal{F}$  is a constructible sheaf of  $\mathbb{Z}/N\mathbb{Z}$ -modules there are isomorphisms*

$$\epsilon^* Rf_*^{et,i} \mathcal{F} \cong Rf_*^{an,i} \epsilon^* \mathcal{F}$$

$$\epsilon^* Rf_!^{et,i} \mathcal{F} \cong Rf_!^{an,i} \epsilon^* \mathcal{F}$$

Fix a prime  $\ell$ . A constructible  $\ell$ -adic sheaf is a system  $\dots \mathcal{F}_n \rightarrow \mathcal{F}_{n-1} \dots$  of sheaves on  $X_{et}$ , such that each  $\mathcal{F}_n$  is a constructible  $\mathbb{Z}/\ell^n\mathbb{Z}$ -module and the maps induce isomorphisms  $\mathcal{F}_n \otimes \mathbb{Z}/\ell^{n-1}\mathbb{Z} \cong \mathcal{F}_{n-1}$ . Standard sheaf theoretic operations can essentially be defined componentwise, and they work as expected [SGA5, SGA4h]. Given a constructible sheaf  $\mathcal{F} = \{\mathcal{F}_n\}$  on  $X_{et}$ , define

$$\epsilon^* \mathcal{F} = \varprojlim_n \epsilon^* \mathcal{F}_n$$

$$\epsilon^*(\mathcal{F} \otimes \mathbb{Q}_\ell) = (\varprojlim_n \epsilon^* \mathcal{F}_n) \otimes \mathbb{Q}_\ell$$

on  $X_{cl}$ . Then with this notation, the above theorem extends to the case where  $\mathcal{F}$  is an  $\ell$ -adic sheaf  $(\otimes \mathbb{Q}_\ell)$  [BBD, §6.1].

## C Classical $t$ -structure for mixed Hodge modules

Saito [S1, S2] has introduced a category of mixed Hodge modules<sup>1</sup>  $MHM(S)$ . When  $S$  is nonsingular, an object  $\mathcal{M}$  of this category consists of a filtered perverse sheaf  $(K, W)$  of  $\mathbb{Q}$ -vector spaces on  $S^{an}$  together with compatible bifiltered regular holonomic  $D_S$ -module  $(M, W, F)$ . These are subject to a rather delicate set of conditions that we will not attempt to spell out. The definition is inductive. In particular, when  $S$  is a point,  $MHM(S)$  is nothing but the

<sup>1</sup>To avoid confusion, we note that we are following the conventions of §4 of [S2].

category of polarizable mixed Hodge structures. One has a forgetful functor  $rat : MHM(S) \rightarrow Perv(S^{an})$  to the category of perverse sheaves given by  $\mathcal{M} \mapsto K$ . Saito has established the following properties:

1. There is an exact faithful functor  $rat : MHM(S) \rightarrow Perv(S^{an})$  for any  $S$ . It extends to a triangulated functor  $D^bMHM(S) \rightarrow D^bCons(S^{an})$ .
2.  $MHM(S)$  contains the category of admissible variations of mixed Hodge structure. In fact,  $M$  is a variation if and only if  $rat(M)$  is a local system up to shift.
3. Standard sheaf theoretic operations extend to  $D^bMHM(S)$  including Grothendieck's "six operations" and vanishing cycles functors. These are compatible with the corresponding operations on  $D^bCons(S^{an})$  via  $rat$ .

The most natural  $t$ -structure on  $D^bMHM(S)$  has  $MHM(S)$  as its heart. This corresponds to the perverse  $t$ -structure on the constructible derived category, so we refer to this as the perverse  $t$ -structure on  $MHM$ . Saito [S2, remark 4.6] has pointed out that there this a second  $t$ -structure that we call the classical  $t$ -structure which lifts the standard  $t$ -structure on  $D^bCons(S^{an})$  with  $Cons(S^{an})$  as its heart.

**Theorem C.0.12.** *There exists a nondegenerate  $t$ -structure  $({}^cD^{\leq 0}, {}^cD^{\geq 0})$  on  $D^bMHM(S)$  which is compatible with the standard  $t$ -structure on  $D^bCons(S^{an})$ .*

*Proof.* Let  $i_x : x \rightarrow X$  denote the inclusion of a point. Define  $M \in Ob({}^cD^{\leq 0})$  (respectively  $\in Ob({}^cD^{\geq 0})$ ) if  $i_x^*M = 0$  for all  $x \in X$  and  $k > 0$  (respectively  $k < 0$ ). To see that this is a  $t$ -structure, note that it is enough to check this on each step of the filtered union  $D^bMHM(S) = \bigcup D^bMHM(S, \Sigma)$ , where  $MHM(S, \Sigma) \subset MHM(S)$  denotes the full subcategory consisting of mixed Hodge modules such that  $rat(\mathcal{M})$  is  $\Sigma$ -constructible. One can now use induction on the cardinality  $|\Sigma|$ . If  $|\Sigma| = 1$ , the purported  $t$ -structure is in fact what it is claimed to be since it is the perverse  $t$ -structure up to shift. When  $|\Sigma| > 1$ , let  $T$  be a closed stratum and  $U = S - T$ . By induction,  $({}^cD^{\leq 0}, {}^cD^{\geq 0})$  determine  $t$ -structures on  $T$  and  $U$ . For  $S$  this follows by verifying the conditions of [BBD, thm 1.4.10] using [S2, 4.4.1].

$({}^cD^{\leq 0}, {}^cD^{\geq 0})$  is clearly compatible with the standard  $t$ -structure on  $D^bCons(S^{an})$ . □

Let  $Cons-MHM(S)$  denote the heart  ${}^cD^{\leq 0}MHM(S) \cap {}^cD^{\geq 0}MHM(S)$ , and likewise for  $Cons-MHM(S, \Sigma)$ .

**Lemma C.0.13.** *The functor*

$$rat : Cons-MHM(S, \Sigma) \rightarrow Cons(S^{an}, \Sigma)$$

*yields an exact faithful embedding.*

*Proof.* Exactness is already clear from the theorem, so the only issue is faithfulness. This can be proved by induction on  $|\Sigma|$ . This holds when  $|\Sigma| = 1$  because  $\text{Cons-MHM}(S, \Sigma)$  is the category of variations of Hodge structures. In general, let  $i : T \rightarrow S$  be a closed stratum and  $j : U \rightarrow S$  be the complement. Suppose that  $f \in \text{Hom}(M, N)$  is morphism in  $\text{Cons-MHM}(S, \Sigma)$  such that  $\text{rat}(f) = 0$ . We need to prove that  $f = 0$ . By induction  $f|_T = 0$  and  $f|_U = 0$ . From the distinguished triangle

$$j_!j^*M \rightarrow M \rightarrow i_*i^*M \rightarrow j_!j^*M[1]$$

and adjointness we obtain an exact sequence

$$\begin{array}{ccccc} \text{Hom}(i_*i^*M, N) & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(j_!j^*M, N) \\ \downarrow \parallel & & & & \downarrow \parallel \\ \text{Hom}(i^*M, i^*N) & & & & \text{Hom}(j^*M, j^*N) \end{array}$$

Therefore  $f = 0$ . □

## References

- [A1] Y. André, *Pour une théorie inconditionnelle de motifs*, Inst. Hautes Études Sci. Publ. Math. No. 83 (1996)
- [A2] Y. André, *Une introduction aux motifs*, Soc. Math. France (2004)
- [Ar] D. Arapura, *The Leray spectral sequence is motivic*, Invent. Math. 160 (2005)
- [Ar2] D. Arapura, *The Hodge theoretic fundamental group and its cohomology*. The geometry of algebraic cycles, 322, Clay Math. Proc., 9, AMS( 2010)
- [AD] D. Arapura, A. Dhillon, *The motive of the moduli stack of G-bundles over the universal curve*, Proc. Ind. Acad. Sci 118 (2008)
- [AK1] D. Arapura, M. Kumar, *Beilinson-Hodge cycles on semiabelian varieties*, Math. Res. Lett. 16 (2009)
- [SGA4] M. Artin, A. Grothendieck, J-L Verdier, *Theorie de topos et cohomologie étale de schémas* Springer LNM 269, 270, 305 (1972-1973)
- [BFM] P. Baum, W. Fulton, R. Macpherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. No. 45 (1975)
- [BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers* Asterique 100 (1982)
- [B1] A. Beilinson, *Notes on absolute Hodge cohomology*, Appl. Alg. K-theory to Alg. Geom, Comtemp. Math, AMS (1986)

- [B2] A. Beilinson, *On the derived category of perverse sheaves*, *K-theory, arithmetic and geometry*, Springer 1289 (1987)
- [Bl] S. Bloch, *Algebraic cycles and higher K-theory*, *Adv. Math.* 61 (1986)
- [Br] A. Bruguières, *On a tannakian theorem due to Nori*, preprint (2004)
- [C] M. de Cataldo, *The standard filtration on cohomology with compact supports...*, *Contemp. Math.* 496, AMS (2009)
- [CM] M. de Cataldo, L. Migliorini, *The perverse filtration and the Lefschetz hyperplane theorem*, *Annals of Math* 171 (2010)
- [CD] D-C. Cisinski, F. Déglise, *Triangulated categories of mixed motives*, arXiv:0912.2110
- [Dg] F. Déglise, *Finite correspondences and transfers over a regular base*, *Algebraic cycles and motives*, *Lond. Math. Soc. LN* 343, (2007)
- [De1] P. Deligne, *Theorie de Hodge II,III*, *Inst. Hautes Études Sci. Publ. Math* 40, 44, (1971,1974)
- [De2] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, *Galois groups over  $\mathbb{Q}$* , MSRI 16, Springer (1987)
- [De3] P. Deligne, *Catégories tannakiennes*, *Grothendieck Festschrift*, Birkhauser (1990)
- [SGA4h] P. Deligne et. al., *Cohomologie Etale, SGA $_4^{\frac{1}{2}}$*  Springer LNM 569 (1977)
- [DMOS] P. Deligne, J. Milne, A. Ogus, K. Shih, *Hodge cycles, motives, and Shimura varieties*, Springer LNM 900 (1982)
- [F] J. Franke, *On the Brown representability theorem for triangulated categories*, *Topology* 40 (2001)
- [Gi] J. Giraud, *Cohomologie non abelienne*, Springer-Verlag (1971)
- [Gd] R. Godement, *Topologie algebrique et theorie des Faisceaux*, Hermann (1958)
- [Gr] P. Griffiths, *Hodge theory and geometry*, *Bull. Lond. Math. Soc.* 36 (2004)
- [SGA5] A. Grothendieck, et. al. *Cohomologie  $\ell$ -adique et Fonctions L*, Springer LNM 589, (1977)
- [I] B. Iversen, *Cohomology of sheaves*, Springer-Verlag (1986)
- [J1] U. Jannsen, *Mixed motives and algebraic K-theory* Springer LNM 1400 (1990)

- [J2] U. Jannsen, *Motives, numerical equivalence and semisimplicity*, Invent Math. 107 (1991)
- [Jo] J.P. Jouanolou, *Un suit exacte de Mayer-Vietoris en K-thorie algebrique*, in Algebraic K-theory, Lect. Notes Math 341, Springer-Verlag (1973)
- [K] S. Kleiman, *Algebraic cycles and the Weil conjectures* Dix Exposés, North-Holland (1968)
- [JS] A. Joyal, R. Street, *Introduction to Tannaka duality and quantum groups*, in Springer LNM 1488 (1991)
- [K] M Kashiwara, *A study of a variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. 22 (1986)
- [KS] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Springer-Verlag (2006)
- [LM] G. Laumon, L. Moret-Bailly, *Champ Algebriques*, Springer-Verlag (2000)
- [L1] M. Levine, *Mixed Motives*, Math. Surveys 57, AMS (1998)
- [L2] M. Levine, *Mixed Motives*, Handbook of K-theory, Springer-Verlag (2005)
- [Li1] J. Lipman, *Equivsingularity and simultaneous resolution of singularities*, Resolution of singularities (Obergrurgl 1997), Birkhäuser, (2000)
- [Li2] J. Lipman, *Notes on derived functors and Grothendieck duality*, SLN 1960, Springer-Verlag (2009)
- [M] S. MacLane, *Categories of the working mathematician, 2nd ed.* Springer-Verlag (1998)
- [Mr] J. Mather, *Notes on topological stability*, Harvard (1970)
- [Mi] J. Milne, *Etale cohomology*, Princeton U. Press (1980)
- [N] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*. J. Amer. Math. Soc. 9 (1996)
- [N1] M. Nori, *Constructible sheaves*, Proc. Int. Conf. on Algebra, Mumbai (2000), TIFR (2002)
- [N2] M. Nori, *Unpublished notes on motives*
- [PS] C. Peters, J. Steenbrink, *Mixed Hodge structures*, Springer-Verlag (2008)
- [Sv] N. Saavedra, *Catégories Tannakiennes* LNM 265, Springer (1972)
- [S1] M. Saito, *Module Hodge polarizables*, Publ. Res. Inst. Math. Sci. 25 (1989)

- [S2] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990)
- [S3] M. Saito, *Arithmetic mixed sheaves*, Invent Math 44 (2001)
- [SZ] J. Steenbrink, S. Zucker, *Variations of mixed Hodge structures*, Invent. Math. 80 (1983)
- [T] B. Teissier, *Varietes Polaires II*, pp 314-491, LNM 961, Springer
- [V1] J-L. Verdier, *Class d'homologies d'un cycle*, Sem. Geom. An. Astérisque no. 36-37, 101-151 (1976)
- [V2] J-L. Verdier, *Stratification de Whitney et théor'eme de Bertini-Sard*. Invent 36 (1976)
- [Vi] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry. Grothendieck's FGA explained. AMS (2005)
- [VSF] V. Voevodsky, A. Suslin, E. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. Math Stud. 143, Princeton (2000)

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